11/05/2020

Span Set

let S be a non-empty (may be infinite) subset of a vector space V. then the linear span, or span(S) in V is the set of all possible (finite) linear combination of the vector of S

i.e

$$span(S) = \{ c_1 v_1 + c_2 v_2 \cdots + c_n v_n : c_1 c_2 \cdots c_n \in \mathbb{R}$$
$$v_1, v_2 \cdots v_n \in S \} \text{ and}$$
$$c_1 v_1 + c_2 v_2 \cdots + c_n v_n = u \quad \forall u \in V$$

Imp

if every vector in V is the linear combination of vectors of S $\Rightarrow$ span(S) = V.

Qn $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ S is subset of $\mathbb{R}^3$ To show span(S) $= \mathbb{R}^3$

Soln we know that span(S) $= \mathbb{R}^3$

$\Rightarrow$ every vector in $\mathbb{R}^3$ is the linear combination of vectors of S then span(S) $= \mathbb{R}^3$

let $u \in \mathbb{R}^3$ $u = (a_1, a_2, a_2)$ arbitrary

$a_1, a_2, a_3 \in \mathbb{R}$

$$u = (a_1, a_2, a_3) = (a_1, 0, 0) + (0, a_2, 0) + (0, 0, a_3)$$
$$u = a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1)$$
$$a_1 a_2 a_3 \in \mathbb{R}$$

$\therefore v$ is the linear combination of vector in $S = \{(1,0,0), (0,1,0), (0,0,1)\}$

$\because v$ is arbitrary

$\therefore$ span$(S) = \mathbb{R}^3$

Imp method for simplifying span(S) Using Row Reduction

Suppose that S is finite subset of $\mathbb{R}^n$ containing k vectors $k \geq 2$

Step 1 ~~By using the vectors~~ k

Step 1 form $k \times n$ matrix A, by using the vectors in S as the row of A

Step 2 find reduced row echelon form A called matrix B

Step 3 A simplified form for span(S) is given by the set of all linear combination of the non-zero row of B

Ex $S = \{(1,-1,1), (2,-3,3), (0,1,-1)\} \subseteq \mathbb{R}^3$
Use the simplified span method to find span(S) in simplified form
[DU CSE-2 2018]

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Soln $S = \{(1,-1,1), (2,-3,3), (0,1,-1)\}$

Step 1 form a matrix A by using the vectors in S as the rows of A

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

Step 2 Row reduce A

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 3 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$ → this is reduced row echelon form A

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 3 span(s) is the set of all linear combination of the non-zero row of C

$\text{span}(S) = \{a(1,0,0) + b(0,1,-1) \; a,b \in \mathbb{R}\}$

Qn $S = \{(1,3,0,1), (0,0,1,1), (0,1,0,2)$
$(1,5,1,4)\}$ $S \subseteq \mathbb{R}^4$

use simplified span method to find span (S) in simplified form prove that the set

$S = \{ (1,3,-1), (2,7,-3), (4,8,-7) \}$

$\text{span}(S) = \mathbb{R}^3$

use the simplified span method prove that $\text{span}(S) = \mathbb{R}^3$

form matrix A By using vectors of S

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 7 & -3 \\ 4 & 8 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim$ reduced row echelon form of A

$\Rightarrow \text{span}(S) = \{ a(1,0,0) + b(0,1,0) + c(0,0,1) : a, b, c \in \mathbb{R}^3$

$= \{ (a,b,c) : a,b,c \in \mathbb{R} \} = \mathbb{R}^3$

$\Rightarrow \text{span}(S) = \mathbb{R}^3$

Prove that set

$S = \{ (1,-2,2), (3,-4,-1), (1,-4,9), (0,2,-7$

$\text{span}(S) \neq \mathbb{R}^3$

Qn $S = \{ 5 - x + x^3,\ 10 - 3x + 3x^3,\ -6 - 5x + 5x^3,\ 6x - 6x^3 - 13 \}$ is subset of $P_3(\mathbb{R})$

find span(S) = ?

Soln $\because$ S is the set of poly of degree 3

$\Rightarrow a_0 + a_1 x + a_2 x^2 + a_3 x^3 \in S$

form matrix A from S

$$\begin{bmatrix} 5 & -1 & 0 & 1 \\ 10 & -3 & 0 & 3 \\ -6 & -5 & 0 & 5 \\ -13 & 6 & 0 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{span}(S) = \{ a(1-x^2) + b(x^3) : a\ b \in \mathbb{R} \}$

$= \{ a - ax^2 + bx^3 : a, b \in \mathbb{R} \}$

Qn $S = \left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

find span(S)

Soln form matrix A By using ~~matrix~~ A S

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\text{span}(S) = \left\{ a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & -b \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & -c \end{bmatrix} : a, b, c \in \mathbb{R} \right.$$

$$= \left\{ \begin{bmatrix} a & b \\ c & -a-b-c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

Qn $S = \left\{ \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} -2 & -5 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ -3 & 4 \end{bmatrix} \right\}$

find span(S) = ?

Qn $S = \left\{ \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -5 & 0 \\ 0 & 12 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \right\}$

find span(S) = ?

Def^n Let $S = \{v_1, v_2 \cdots v_n\}$ be a finite non-empty subset of a (real) vector space V Then S is said to be linearly dependent iff $\exists$ real number $c_1, c_2, \cdots c_n$ not all zero such that

$c_1v_1 + c_2v_2 - + c_nv_n = 0$

i.e one of $c_1\ c_2 \cdots c_n$ not zero

Def^n Let $S = \{v_1, v_2 - v_n\}$ be a finite non-empty subset of a vector space V. then S is said to be linearly-independent if for any set of real number $a_1\ a_2 \cdots a_n$ s.t

$a_1v_1 + a_2v_2 \cdots + a_nv_n = 0$

$\Rightarrow a_1 = a_2 - = a_n = 0$

Result $S = \{v\}$ is linearly dependent iff $v = 0$

Result $S = \{v\}$ is linearly Independent iff $v \neq 0$

Theo Set $S = \{v_1 v_2\}$ is L.D iff atleast one of the vectors is scalar multiple of the other.

proof Let $S = \{v_1 v_2\}$ is linearly dependent

$\Rightarrow \exists$ scalar $a_1 a_2 \in$ (not ~~of~~ both zero)

Let $a_1 \neq 0$ & $a_1v_1 + a_2v_2 = 0$

$\because a_1 \neq 0 \Rightarrow v_1 = -\frac{a_2}{a_1} v_2$ (possible)

$\Rightarrow v_1$ is scalar multiple of v_2

Conversely let v_1 is scalar multiple of v_2

$\Rightarrow v_1 = \alpha v_2$ for some $\alpha \in \mathbb{R}$

$\Rightarrow v_1 - \alpha v_2 = 0 \Rightarrow 1 \cdot v_1 + (-\alpha) \cdot v_2 = 0$

$\because$ coeff of v_1 is $1 \neq 0$

By defn $S = \{v_1, v_2\}$ is L.D

Any finite subset of a vector space that contain zero is L.D

Ex $S = \{(0,0), (1,2)\}$ S is L.D

Test for Linear Independence By using row reduction

Step 1 form the matrix A By using the vector in S as colm of A

Step 2 Row reduce A to obtain B

Step 3 If each colm has pivot in row reduce matrix then A is Linearly ~~Idep~~ Independent

Step (4) If Any colm has no pivot ⇒ A is L.D

Qn $S = \{(1,-1,0,2), (0,-2,1,0), (2,0,-1,1)\}$ of $\mathbb{R}^4$ S is LI or not.

Soln Step 1 form matrix A by using the vector in S as colm of A

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & -2 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since each colm has pivot

⇒ S is Linearly Independent

Qn $S = \{(1,2,-1), (1,-2,1), (-3,2,-1)\}$ S is L.I or L.D?

Qn $S = \{(1,-1,0,2), (0,-2,1,0), (2,0,-1,1)\}$ S is L.D or L.I

Form a matrix A of $n \times n$ order from vectors of set S

① $|A| = 0$ iff set is L.D

② $|A| \neq 0$ iff set is L.I

Qn $S = \{(3,1,-1), (5,-2,2), (2,2,-1)\}$
S is L.I or L.D. over $\mathbb{R}^3$

Soln form matrix A

$$A = \begin{bmatrix} 3 & 5 & 2 \\ 1 & -2 & 2 \\ -1 & 2 & -1 \end{bmatrix} \Rightarrow |A| = -8 \neq 0$$

$\Rightarrow$ S is L.I

Qn $S = \{1+x+x^2, -1+x^2, 1+x^2\} \subseteq P_2(\mathbb{R})$
S is L.I or not

Soln form matrix A from set S

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$1+x+x^2 \to (1,1,1)$
$-1+0\cdot x+x^2 \to (-1,0,1)$
$1+0x+x^2 \to (1,0,1)$

$|A| = 2 \neq 0 \Rightarrow$ S is L.I

Qn $S = \left\{ \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ -5 & 2 \end{pmatrix}, \begin{pmatrix} 4 & -1 \\ -5 & 2 \end{pmatrix}, \begin{pmatrix} 3 & -3 \\ 0 & 0 \end{pmatrix} \right\}$
S is L.I or not

Soln form matrix A from set S

$$A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ -2 & 2 & -1 & -3 \\ 0 & -6 & -5 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix}_{4\times 4}$$

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$$\begin{bmatrix} 1 & 3 & 4 & 3 \\ -2 & 2 & -1 & -3 \\ 0 & -6 & -5 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -21/2 \\ 0 & 1 & 0 & -15/2 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

$\Rightarrow$ B has no pivot in colm 4

$\Rightarrow$ S is L.D.

Qn $S = \left\{ \begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} -2 & -5 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ -3 & 4 \end{pmatrix} \right\}$

S is L.I or L.D.

An infinte subset S of vector space V is said to be L.I if every finite subset S is L.I

An infinite subset S of a vector space V is said to be L.D if there is some finite subset T of S which is L.D

Qn $S = \left\{ \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 6 & -1 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} -11 & 3 \\ -2 & 2 \end{pmatrix} \right\}$

S is L.D or L.I ?

Lecture - 49

{Bases for a vector space}

Defⁿ: Let V be a vector space and let S be subset of V. then S is called a bases for V if S is Linearly Independent and span(S) = V

i.e S is L.I and span(S) = V.

Qn Show that S = {(1,0,0), (0,1,0), (0,0,1)} S is bases for $\mathbb{R}^3$

Solⁿ for bases we shall prove S is L.I & span(S) = $\mathbb{R}^3$

form matrix A from set for L.I

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

∴ each colm of A has pivot ⇒ S is L.I

form matrix A from set A

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\text{span}(S) = \{ a(1,0,0) + b(0,1,0) + c(0,0,1) : a, b, c \in \mathbb{R} \}$

$= \{ (a,0,0) + (0,b,0) + (0,0,c) : a, b, c \in \mathbb{R}$

$= \{ (a,b,c) : a, b, c \in \mathbb{R} \} = \mathbb{R}^3$

$\Rightarrow \text{span}(S) = \mathbb{R}^3$

$\Rightarrow$ S is L.I & $\text{span}(S) = \mathbb{R}^3$

Hence $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ is bases for $V = \mathbb{R}^3$

Qn $S = \{1, x, x^2, x^3\}$ is bases for $P_3(\mathbb{R})$

Sol^n we shall check two condition

(i) S is L.I (ii) $\text{span}(S) = P_3(\mathbb{R})$

Let ~~$a_1, a_2, a_3, a_4 \in \mathbb{R}$ be scalars s.t~~

~~$a_1(1) + a_2(x) + a$~~

$1 \longrightarrow 1 + 0\cdot x + 0\cdot x^2 + 0\cdot x^3 \sim (1,0,0,0)$

$x \longrightarrow 0 + 1\cdot x + 0\cdot x^2 + 0\cdot x^3 \sim (0,1,0,0)$

$x^2 \longrightarrow 0 + 0\cdot x + 1\cdot x^2 + 0\cdot x^3 \sim (0,0,1,0)$

$x^3 \longrightarrow 0 + 0\cdot x + 0\cdot x^2 + 1\cdot x^3 \rightarrow (0,0,0,1)$

for L.I form matrix A

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}$$

$\because$ each colm has pivot $\Rightarrow$ set is L.I

$\Rightarrow S = \{1, x, x^2, x^3\}$ is L.I

for span(S) form matrix A

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \simeq \text{Row Reduced echelon form}$$

$$\text{span}(S) = \{a(1,0,0,0) + b(0,1,0,0) + c(0,0,1,0) + d(0,0,0,1) : a,b,c,d \in \mathbb{R}\}$$

$$\simeq \{a(1+0\cdot x+0\cdot x^2+0\cdot x^3) + b(0+1\cdot x+0\cdot x^2+0\cdot x^3) + c(0+0\cdot x+1\cdot x^2+0\cdot x^3) + d(0+0\cdot x+0\cdot x^2+1\cdot x^3) : a,b,c,d \in \mathbb{R}\}$$

$$= \{a + bx + cx^2 + dx^3 : a,b,c,d \in \mathbb{R}\} = P_3(\mathbb{R})$$

$\Rightarrow$ span$(S) = P_3(\mathbb{R})$

$\because$ S is L.I & span$(S) = P_3(\mathbb{R}) \Rightarrow$ Set S is bases for $P_3(\mathbb{R})$

Qn Prove that $S = \{(1,3,-1), (2,7,-3), (4,8,-7)\}$ spans $\mathbb{R}^3$. Examine whether S form a bases for $\mathbb{R}^3$ [DU GE-2 2016]

Sol^n form matrix A from set S

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 7 & 8 \\ -1 & -3 & -7 \end{bmatrix} \quad |A| = -9 \neq 0$$

$\Rightarrow$ S is L.I

for span(S) Again form matrix A row we

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 7 & -3 \\ 4 & 8 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ B is the row reduce echelon form

$\Rightarrow$ span(S) is the linear combination of row of B

$$\text{span}(S) = \{a(1,0,0) + b(0,1,0) + c(0,0,1) ; a, b, c \in \mathbb{R}\}$$

$$= \{(a,0,0) + (0,b,0) + (0,0,c) ; a,b,c \in \mathbb{R}$$

$$= \{(a,b,c) : a,b,c \in \mathbb{R}\} = \mathbb{R}^3$$

$\Rightarrow$ span(S) $= \mathbb{R}^3$

$\therefore$ S is L.I & span(S) $= \mathbb{R}^3 \Rightarrow$ S is basis for $V = \mathbb{R}^3$

Qn $S = \{(1,4,2,0), (0,2,1,0), (-3,1,-1,0)$, $(5,-2,0,-3)\}$
S is basis for $\mathbb{R}^3$?

Qn $S = \left\{ \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 2 & -2 \end{pmatrix}, \begin{pmatrix} 3 & 7 \\ 0 & -2 \end{pmatrix} \right.$

S is basis for $M_{2\times 2}$?

Qn $S = \{1+x^2, -1+x^2, 1+x+x^3\}$
S is basis for $P_3(\mathbb{R})$

Dimension: Dim is the number of elements or vectors in any basis for V

Defn finite-dim vector space: A vector space V is said to be finite-dim if its ~~basis~~ basis containing a finite number of elements

Defn A vector space V is called infinite-dim if its basis ~~cont~~ is not finite