Lecture - 22

Some properties as lim $x \to \infty, -\infty$

If M, L & K are real numbers & and

$\lim_{x \to \pm\infty} f(x) = L$, $\lim_{x \to \pm\infty} g(x) = M$ then

(1) $\lim_{x \to \pm\infty} (f(x) \pm g(x)) = L \pm M$

(2) $\lim_{x \to \pm\infty} (f(x) \cdot g(x)) = L \cdot M$

(3) $\lim_{x \to \pm\infty} (K f(x)) = K \cdot L$, $\lim_{x \to \pm\infty} (K \cdot g(x)) = K$

(4) $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = \frac{L}{M}$ $M \neq 0$

(5) Imp Power rule: If r, s are integers with no common factor $s \neq 0$ then

$\lim_{x \to \pm\infty} (f(x))^{r/s} = (\lim_{x \to \pm\infty} f(x))^{r/s}$

$= L^{r/s}$

Thm If A is a real number and r is a positive rational number. then

(i) $\lim_{x \to +\infty} \frac{A}{x^r} = 0$

(ii) If r is such that x^r defined for $x<0$ then $\lim_{x \to -\infty} \frac{A}{x^r} = 0$

Proof (i) To show $\lim_{x \to +\infty} \frac{A}{x^r} = 0$

for this firstly we shall prove

$\lim_{x \to +\infty} \frac{1}{x} = 0$

let $\epsilon > 0$ be given we must find M s.t

$|f(x) - L| < \epsilon$ whenever $x > M$

$\Rightarrow$ $|\frac{1}{x} - 0| < \epsilon$ whenever $x > M$

$\Rightarrow \frac{1}{x} < \epsilon$ $\quad \because [\,|\frac{1}{x}| = \frac{1}{x}$ if $x>0]$

$\Rightarrow x > 1/\epsilon$

choose $M = 1/\epsilon$

$\Rightarrow$ If $x > M = 1/\epsilon \Rightarrow |f(x) - L| < \epsilon$

$\Rightarrow \lim_{x \to \infty} \frac{A}{x^r} = 0$

Need to show $\lim_{x \to \infty} \frac{A}{x^r} = 0$

$\because$ r is rational number $\Rightarrow r = p/q$
$p, q \in I$ $q \neq 0$ with $(p, q) = 1$

Now $\lim_{x \to \infty} \frac{A}{x^r} = \lim_{x \to +\infty} \frac{A}{x^{p/q}}$

$= A \left[\lim_{x \to +\infty} \frac{1}{x^{p/q}} \right]$

Now By power rule

$\lim_{x \to \pm\infty} (f(x))^{r/s} = \left[\lim_{x \to \pm\infty} f(x) \right]^{r/s}$

where r, s has no common factor

$\cancel{A \left[\lim_{x \to +\infty} \frac{1}{\ } \right.} = A \left[\lim_{x \to +\infty} \frac{1}{x} \right]^{p/q}$

$= A \cdot [0]^{p/q}$ $\left| \because \cancel{\lim_{x \to +\infty}} \lim_{x \to +\infty} \frac{1}{x} = 0 \right|$

$= A \cdot 0 = 0$

$\Rightarrow \lim_{x \to +\infty} \frac{A}{x^r} = 0$

Qn $\lim_{x \to \infty} \left(\frac{x+1}{2x+2}\right)^{1/8} = ?$

Sol

Result if $h(x) = f(x) \cdot g(x)$ be a f^n with either one of $f(x)$ or $g(x)$ is bounded and one of $\lim_{x \to 0} f(x) = 0$ or $\lim_{x \to 0} g(x) = 0$

$\Rightarrow \lim_{x \to 0} f(x) \cdot g(x) = 0$

Qn $\lim_{x \to \infty} e^{-x} \cos x = ?$

let $x = \frac{1}{t}$ $\therefore$ $x \to \infty \Rightarrow \frac{1}{t} \to \infty$

$\Rightarrow t \to 0$

$\Rightarrow$ Replace x by $\frac{1}{t}$ $\Rightarrow$ limit will be chang. as $x \to \infty \Rightarrow t \to 0$

$\lim_{x \to \infty} e^{-x} \cos x = \lim_{t \to 0} e^{-1/t} \cos 1/t$

$= \lim_{t \to 0} \frac{\cos(1/t)}{e^{-1/t}}$ — (i)

$\because$ $\cos x$ is bounded $\forall x$

$\Rightarrow$ $\cos\left(\frac{1}{x}\right)$ is also bounded $\forall x$

$\Rightarrow$ $\cos\left(\frac{1}{t}\right)$ is also bounded $\forall$ ~~x~~ t

$\Rightarrow$ let $f(t) = \cos\left(\frac{1}{t}\right)$ bounded $\forall t$

& $g(t) = e^{-1/t}$

$\because$ $\lim\limits_{t \to 0} g(t) = \lim\limits_{t \to 0} e^{-1/t} = e^{-\infty} = 0$

and $f(t) = \cos(1/t)$ is bounded

$\Rightarrow$ By Last Result

~~$\lim\limits_{x \to}$~~ $\lim\limits_{t \to 0} f(t).g(t) = \lim\limits_{t \to 0} e^{-1/t}.\cos\left(\frac{1}{t}\right)$

$= 0$

$\therefore$ $\lim\limits_{x \to \infty} e^{-x}\cos x = \lim\limits_{t \to 0} e^{-1/t}.\cos\left(\frac{1}{t}\right)$

$\Rightarrow$ $\lim\limits_{x \to \infty} e^{-x}\cos x = 0$

Sandwich theorem for infinite limit

If $g(x) \le f(x) \le h(x)$ be a fn and

$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$

then $\lim_{x \to a} f(x) = L$ when ~~$a \to \infty$~~ or $a = -\infty, \infty$

Qn $\lim_{x \to \infty} \frac{\sin x}{x} = ?$

$\because$ $\sin x$ is bounded fn $\forall x$
and $-1 \le \sin x \le 1$

let $x \neq 0$

$\Rightarrow -\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$ —(I)

By using squeeze theorem let
$g(x) \le f(x) \le h(x)$ —(II)

and $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = l$

$\Rightarrow \lim_{x \to a} f(x) = l$

from (I) & (II) we get

$g(x) = -1/x$ $f(x) = \frac{\sin x}{x}$ and $h(x) = \frac{1}{x}$

$\therefore \lim_{x\to\infty} g(x) = \lim_{x\to\infty}\left(-\frac{1}{x}\right) = 0$

$\therefore \lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{1}{x} = 0$

By Squeez theorem

$\lim_{x\to\infty} \frac{\sin x}{x} = 0$

Qn $\lim_{x\to\infty} \frac{\cos x^2}{x^3} = ?$ $\frac{\cos x^2}{x^3} = f(x)$ $x \neq 0$

Qn $\lim_{x\to\infty} x \cdot \sin\frac{1}{x} = ?$ $x \neq 0$

Sol.n let $x = \frac{1}{t}$ t

$\therefore x \to \infty \Rightarrow \frac{1}{t} \to \infty \Rightarrow t \to 0$

$\Rightarrow \lim_{x\to\infty} x \sin\frac{1}{x} = \lim_{t\to 0}\left(\frac{\sin t}{t}\right)$

$\therefore \lim_{x\to 0} \frac{\sin x}{x} = 0$ [Do your self]

$\Rightarrow \lim_{t\to 0} \frac{\sin t}{t} = \lim_{x\to\infty} x \cdot \sin\frac{1}{x} = 0$