Series

Def 1: $\displaystyle\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$

Def 2: The integral $\displaystyle\int_a^{\infty} f(x)\,dx$ is said to converge if it has a finite valve and diverges if it is $\pm\infty$.

Cauchy's Integral Test :

Let $u(x)$ be a positive, monotonically decreasing and integrable on $[1,\infty[$
For each $n \in \mathbb{N}$, let $u_n = u(n)$
Then the series $\displaystyle\sum_{n=1}^{\infty} u_n$ and the integral $\displaystyle\int_1^{\infty} u(x)\,dx$ either both converge or diverge.

Q: Use integral test to discuss convergence of $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}, p>0$
(p-series)

Sol:

Define $u(x) = \dfrac{1}{x^p}$ on $[1,\infty[$

$u(x)$ is positive, monotonically dec. and integrable on $[1,\infty[$ $\left(\because x^p \uparrow \text{ in } [1,\infty[\quad \dfrac{1}{x^p}\downarrow \right.$

$\therefore$ by the integral test
$\displaystyle\sum_{n=1}^{\infty} u_n$ has same behaviour as $\displaystyle\int_1^{\infty} u(x)\,dx$.

Case 1 : $p=1$

$$\int_1^{\infty} u(x)\,dx = \int_1^{\infty}\frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty}\left[\log x\Big|_1^t\right] = \lim_{t\to\infty}\left[\log t - 0\right] = \infty$$

Consider $I_t = \int_1^{\infty} u(x)\,dx = \int_1^{\infty} \frac{1}{x^p}\,dx$

$$= \lim_{t\to\infty} \int_1^{t} \frac{1}{x^p}\,dx$$

Consider $\int_1^t \frac{1}{x^p}\,dx$

$$= \left.\frac{x^{1-p}}{1-p}\right]_1^t = \frac{1}{1-p}\left[t^{1-p} - 1\right]$$

$$\lim_{t\to\infty} I_t = \begin{cases} \frac{1}{1-p}\,[0-1] & \text{if } p>1 \quad (1-p<0) \\ \text{and} & \\ \frac{1}{1-p}\,[\infty - 1] & \text{if } p<1 \quad (1-p>0) \end{cases}$$

$$\therefore \lim_{t\to\infty} I_t = \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ \infty & \text{if } p<1 \end{cases} \quad \text{——— (2)}$$

From (1) & (2)

$\int_1^{\infty} u(x)\,dx$ converges if $p>1$ & div. if $p\le 1$

$\therefore \sum_{p=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$ and div. if $p\le 1$.