Reduction formulae

A formula by mean of which a given integral can be reduced to some known integral is called a reduction formulae.

The basic technique involved in obtaining a reduction formula is integration by part.

e.g. Find the reduction formulae for

$\int \sin^n x \, dx$ (n being a positive integer)

and hence evaluate $\int_0^{\pi/2} \sin^n x \, dx$

Sol^n we have

$$\int \sin^n x \, dx = \int \sin^{n-1} x \sin x \, dx$$

Integrating by parts.

$$\int \sin^n x \, dx = -\cos x \sin^{n-1} x + (n-1)\int \cos x \sin^{n-2} x \cos x \, dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \, dx - (n-1)\int \sin^n x \, dx$$

$$\therefore (1+n-1)\int \sin^n x \, dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \, dx$$

Hence

$$\boxed{I_n = \int \sin^n x \, dx = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n}\int \sin^{n-2} x \, dx}$$

Now

$$\int_0^{\pi/2} \sin^n x\,dx = \left[-\frac{\cos x \sin^{n-2} x}{n-1}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$= \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$\Rightarrow I_n = \frac{n-1}{n} I_{n-2}$$

Replacing n by $n-2, n-4, \ldots 3, 2,$

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}, \quad I_{n-4} = \frac{n-5}{n-4} I_{n-6}, \ldots$$

$I_3 = \frac{2}{3} I_1$; if n is odd

$I_2 = \frac{1}{2} I_0$; if n is even.

Now

$$I_1 = \int_0^{\pi/2} \sin x\,dx = -\left[\cos x\right]_0^{\pi/2} = 1$$

and $$I_0 = \int_0^{\pi/2} \sin^0 x\,dx = \int_0^{\pi/2} 1 \cdot dx = \pi/2$$

Hence

$$\boxed{\int_0^{\pi/2} \sin^n x\,dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3}, & \text{if } n \text{ is odd} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2}, & \text{if } n \text{ is even} \end{cases}}$$

Q Find the reduction formula for

$\int \cos^n x \, dx$ (n being a positive integer)

hence evaluate $\int_0^{\pi/2} \cos^n x \, dx$

→ Do yourself

Q. Obtain the reduction formula for.

$I_{m,n} = \int \sin^m x \cos^n x \, dx$; where m and n are positive integer.

Soln we have.

$$\int \sin^m x \cos^n x \, dx = \int \sin^{m-1} x (\sin x \cos^n x) \, dx$$

Integrating by parts.

$$\int \sin^m x \cos^n x \, dx = -\frac{\cos^{n+1} x}{n+1} \sin^{m-1} x + \frac{m-1}{n+1} \int \cos^{n+1} x \sin^{m-2} x \cos x \, dx$$

$$= -\frac{\cos^{n+1} x \sin^{m-1} x}{n+1} + \frac{m-1}{n+1} \int \sin^{m-2} x \cos^n x (1 - \sin^2 x) \, dx$$

$$= -\frac{\cos^{n+1} x \sin^{m-1} x}{n+1} + \frac{m-1}{n+1} \int \sin^{m-2} x \cos^n x \, dx -$$

$$\frac{m-1}{n+1} \int \sin^m x \cos^n x \, dx$$

$$\therefore \left(1+\frac{m-1}{n+1}\right)\int \sin^m x \cos^n x \, dx$$

$$= -\frac{\cos^{n+1} x \sin^{m-1} x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^n x \, dx$$

Hence

$$\boxed{\begin{aligned} I_{m,n} &= \int \sin^m x \cos^n x \, dx \\ &= -\frac{\cos^{n+1} x \sin^{m-1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x \, dx \end{aligned}}$$

Q. 1 Obtain a reduction formula for.

$$\int e^{ax} \sin^n x \, dx$$, where n is positive integer.

2 Find reduction formula for.

(a) $\int \cos^m x \cos nx \, dx$

(b) $\int \cos^m x \sin nx \, dx$

(c) $\int \sin^m x \cos nx \, dx$

(d) $\int \sin^m x \sin nx \, dx$

Do yourself.