Maximal linearly Independent Sets & Minimal Spanning Sets

A subset S of a vector space V is said to be maximal linearly Independent subset of V if S is linearly independent & any subset T of V that properly contains S is linearly dependent.

A subset S of a vector space V is said to be minimal spanning subset of V if S spans V & no proper subset of S spans V.

Example :- Let $B = \{[2,3,0,-1], [-1,1,1,-1]\}$. Show that B is a maximal linearly independent subset of $S = \{[1,4,1,-2], [-1,1,1,-1], [3,2,-1,0], [2,3,0,-1]\}$

Soln. The set $B = \{[2,3,0,-1]$, $[-1,1,1,-1]\}$ is linearly independent as none of the vectors is multiple of the other. Now, B is a maximal L.I. subset of S by showing that any subset of S that properly contains B is linearly dependent. The only subsets of S that properly contain B are

$T_1 = \{[2,3,0,-1], [-1,1,1,-1], [1,4,1,-2]\}$

$T_2 = \{$ " " $[3,2,-1,0]\}$

$T_3 = \{$ " " $, [1,4,1,-2], [3,2,-1,0]\} = S$

By using independence Test method, we show that each of the above sets is L.D..

Theo Dimension of a Subspace
Let V be a finite-dimensional V.S. & let W be a subspace of V. Then W is also finite-dimensional with dim(W) ≤ dim(V). Moreover, dim(W) = dim(V) iff W=V.

Ques Find a basis & the dimension of a subspace W of $\mathbb{R}^3$ spanned by set $S = \{[3,2,1], [1,2,0], [-1,2,-1]\}$ of three vectors.

Soln. We first use the Simplified Span Method to find a simplified form for the vectors in span(S). We consider the foll. matrix:-

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 0 \\ -1 & 2 & -1 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & 1 \\ -1 & 2 & -1 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -4 & 1 \\ 0 & 4 & -1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to -\frac{1}{4}R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1/4 \\ 0 & 4 & -1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 4R_2]{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence span(S) is the row space of this row-reduced echelon form.

$$\text{ie } a[1, 0, 1/2] + b[0, 1, -1/4]$$

~~Now~~, $\Rightarrow W = \text{span}(S)$

$$= \text{span}\{[1, 0, 1/2], [0, 1, -1/4]\}$$

Also, these two vectors are linearly independent. Hence, $B = \{[1, 0, 1/2], [0, 1, -1/4]\}$ forms a basis for W.

$\therefore \dim(W) = |B| = 2$.

Ques Find a basis & dimension of the subspace W of $\mathbb{R}^3$ defined by :-

$$W = \{[x, y, z] \in \mathbb{R}^3 : 2x - 3y + z = 0\}$$

Soln. Now, $2x - 3y + z = 0$

$$\Rightarrow x = \frac{3}{2}y - \frac{z}{2}$$

Setting $y = b$ & $z = c$, we have $x = \frac{3}{2}b - \frac{c}{2}$.

$$\therefore W = \left\{\left[\frac{3}{2}b - \frac{c}{2}, b, c\right] : b, c \in \mathbb{R}\right\}$$

$$= \left\{b\left[\frac{3}{2}, 1, 0\right] + c\left[-\frac{1}{2}, 0, 1\right] : b, c \in \mathbb{R}\right.$$

$$= \text{span}\left\{\left[\frac{3}{2}, 1, 0\right], \left[-\frac{1}{2}, 0, 1\right]\right\}$$

$$= \text{span}\left\{[3, 2, 0], [-1, 0, 2]\right\}$$

Also, $[3, 2, 0]$ & $[-1, 0, 2]$ are not multiples of each other. Hence they are L.I.

$\therefore S = \{[3, 2, 0], [-1, 0, 2]\}$ forms a basis for W.

& $\dim(W) = |S| = 2$.