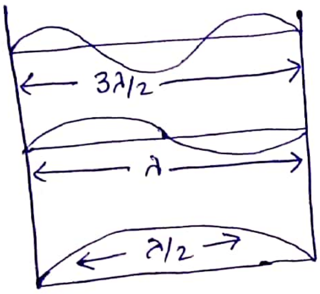
thus

LECTURE — III

In the frequency range ν and $\nu + d\nu$ the energy density can be obtained by multiplying the number of Plank's oscillators lying in that particular range multiplied with the average energy of the plank's oscillator.

So we need to calculate the number of oscillators per unit volume lying in the frequency range ν and $\nu + d\nu$.



if l be the length of the box then allowed wavelength or frequency ν as given by

$$\lambda = \frac{2l}{n} \quad \text{where } n = 1, 2, 3 \ldots$$

$$\nu = \frac{c}{\lambda} = \frac{nc}{\lambda} \quad \text{where } n = 1, 2, 3 \ldots$$

Let us the wave is propagating in 3-D ~~or any rand~~ i.e. any random direction then $l\cos\alpha$, $l\cos\beta$ and $l\cos\gamma$ will be the projections of the edges of the cube on the direction of propogation of the wave.

According to EM theory the allowed waves are these which have nodal points at the faces of cube. Thus the allowed wavelength must satisfy the following condition

$$\lambda = \frac{2l\cos\alpha}{n_1}, \quad \frac{2l\cos\beta}{n_2} \quad \& \quad \frac{2l\cos\gamma}{n_3} \quad \text{——} (9)$$

where n_1, n_2 & n_3 are positive integer.

and We know that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \quad \text{——} (10)$$

then we get from eqn (9) & (10)

$$\frac{n_1^2\lambda^2}{4l^2} + \frac{n_2^2\lambda^2}{4l^2} + \frac{n_3^2\lambda^2}{4l^2} = 1$$

$$n_1^2 + n_2^2 + n_3^2 = \frac{4l^2}{\lambda^2} = \left(\frac{2l}{\lambda}\right)^2 = \left(\frac{2l\nu}{c}\right)^2 \quad \text{——} (11)$$

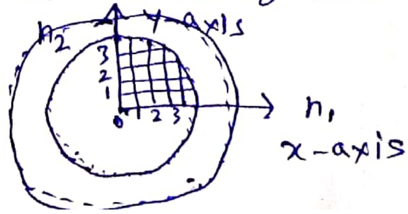
The equation (11) gives the allowed modes of vibration inside the cavity and the ~~total~~ number of possible sets (n_1, n_2, n_3) gives the ~~total~~ no. of mode of vibrations.

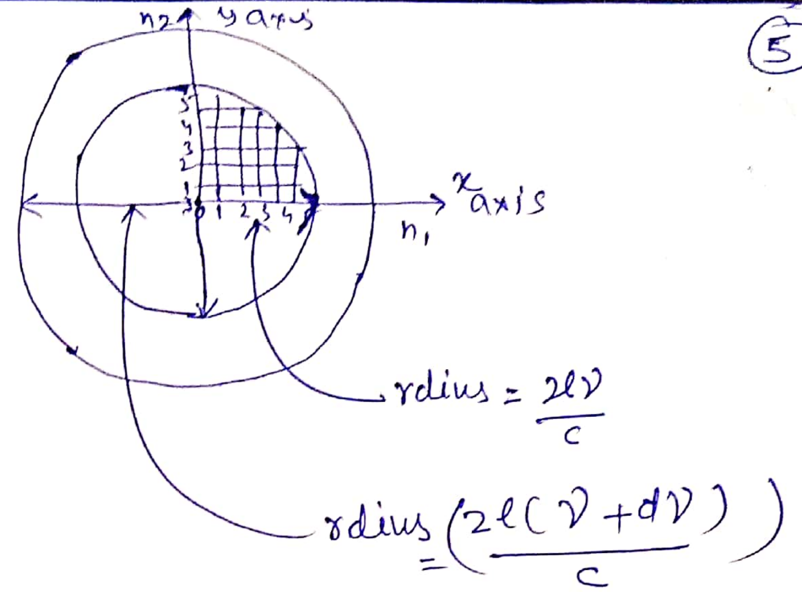
And we find out the total number of mode of vibration with the help of eqn (11)

eqn (11) in - 2D we can rewrite

$$n_1^2 + n_2^2 = \left(\frac{2l\nu}{c}\right)^2 \quad \text{——} (12)$$

if we plot a graph with n_1 along x-axis and n_2 along y-axis then equation (12) represents a circle of radius $\left(\frac{2l\nu}{c}\right)$ as shown in figure.





Hence number of modes of vibration within frequency range ν and $\nu+d\nu$ will be equal to the area under the positive quadrant lying between two circles of radii $\left(\frac{2l\nu}{c}\right)$ and $\frac{2l(\nu+d\nu)}{c}$

Thus the Area is $= \frac{1}{4}\pi\left[\left(\frac{2l(\nu+d\nu)}{c}\right)^2 - \left(\frac{2l\nu}{c}\right)^2\right]$

$$= \frac{\pi}{4} * \frac{8l^2\nu\,d\nu}{c^2}$$

$$= \frac{2l^2\nu\,d\nu}{c^2} \quad \text{———} \quad (13)$$

Now from eqn (13) we calculate the 3-D.

In 3-dimension, number of modes of vibration in the frequency interval ν and $\nu+d\nu$ is given by the $\left(\frac{1}{8}\right)^{th}$ of the volume of the spherical shell within the radii $\frac{2l\nu}{c}$ and $\frac{2l(\nu+d\nu)}{c}$

So the volume $= \frac{1}{8}\left\{\frac{4}{3}\pi\left(\frac{2l(\nu+d\nu)}{c}\right)^3 - \frac{4}{3}\pi\left(\frac{2l\nu}{c}\right)^3\right\}$

$$= \frac{1}{8} \times \frac{4}{3}\pi \times \frac{8l^3}{c^3} \times 3\nu^2 d\nu$$

putting $l^3 = V$, volume of the cube.

$$= \frac{4\pi V \nu^2 d\nu}{c^3}$$

The number of modes per unit volume withen the frequency range ν and $\nu + d\nu$ will be

$$= \frac{4\pi \nu^2}{c^3} d\nu.$$

Since EM waves are transverse in nature so that ~~Mode of~~ number of mode will be double

$$\text{ie} = \frac{8\pi \nu^2}{c^3} d\nu \quad \text{—— (14)}$$

So that number of modes in per unit volume within the frequeny range ν and $\nu + d\nu$ is given by [from eqn (4), (8) & (14)]

$$E_\nu d\nu = \left(\frac{8\pi \nu^2}{c^3} d\nu\right) \frac{h\nu}{\left(e^{h\nu/kT} - 1\right)}$$

$$\boxed{E_\nu d\nu = \frac{8\pi h \nu^3}{c^3} \cdot \frac{1}{\left(e^{h\nu/kT} - 1\right)}} \quad \text{—— (15)}$$

eqn (15) is the Plank's radiation law. its gives the energy density that is energy per unit volume in the frequency range ν and $\nu + d\nu$.