Numerical Method :- By Kapil Kumar

Chapter - 7 Ordinary Differential Equation

Classical Fourth-Order Runge-Kutta Method:- The most popular RK methods are fourth order. As with the second-order approaches, there are an infinite number of versions. The following is the most commonly used form, and we therefore call it the classical fourth-order RK method! -

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

Where $k_1 = f(t_i, y_i)$

$$k_2 = f\left(t_i + \frac{1}{2}h,\ y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(t_i + \frac{1}{2}h,\ y_i + \frac{1}{2}k_2 h\right)$$

$$k_4 = f(t_i + h,\ y_i + k_3 h)$$

Ex). 1.1: Apply R.K. Fourth order to solve the initial value problem and calculate $y(1)$ using $h=1$: $\frac{dy}{dx} = 4e^{0.8t} - 0.5y$, $y(0) = 2$.

Soln:. $k_1 = f(0,2) = 4e^{0.8(0)} - 0.5(2) = 3$

$k_2 = f\left(0 + \frac{1}{2}\times 1,\ 2 + \frac{1}{2}\times 3 \times 1\right) = k_2 = f(0.5, 3.5)$

$k_2 = 4e^{0.8(0.5)} - 0.5(3.5) = 4.217299$

$k_3 = f\left(0 + \frac{1}{2}\times 1,\ 2 + \frac{1}{2}\times 4.217299 \times 1\right) = f(0.5, 4.108649)$

$k_3 = 4e^{0.8(0.5)} - 0.5(4.108649) = 3.912974$

$k_4 = f(0+1,\ 2 + 3.912974 \times 1) = f(1, 5.912974)$

$k_4 = 4e^{0.8(1)} - 0.5(5.912974) = 5.945677$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h = 2 + \frac{1}{6}[3 + 2(4.217299)$
$+ 2(3.912974) + 5.945677] \times 1 = 2 + 4.201037$

$y(1) = 6.201037.$

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Ex:- (2.1): Apply R.K. Fourth order to solve the initial value problem & calculate $y(0.1)$ by using $h = 0.1$ $\frac{dy}{dx} = x^2 - y$, $y(0) = 1$

Soln:

$k_1 = f(x_0, y_0) = f(0,1) = (0)^2 - (1) = -1$

$k_2 = f(0 + \frac{1}{2} \times 0.1, 1 + \frac{1}{2} \times -1 \times 0.1) = f(0.05, 0.95)$

$k_2 = (0.05)^2 - (0.95) = -0.9475$

$k_3 = f(0 + \frac{1}{2} \times 0.1, 1 + \frac{1}{2} \times -0.9475 \times 0.1) = f(0.05, 0.9526)$

$k_3 = (0.05)^2 - (0.9526) = -0.9501$

$k_4 = f(0 + 0.1, 1 + -0.9501 \times 0.1) = f(0.1, 0.9049)$

$k_4 = (0.1)^2 - (0.9049) = -0.8949$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h = 1 + \frac{1}{6}[-1 + 2(-0.9475)$
$+ 2(-0.9501) - 0.8949] \times 0.1 = 1 + (-0.094835)$ ~~= 0.905165~~.
$= 0.905165$. $\therefore y(0.1) = 0.905165$

If we want calculate $y(0.2)$ then we perform 2nd Iteration with

$x_1 = x_0 + h = 0 + 0.1 = 0.1$ & $y_1 = 0.905165$.

<u>Finite Difference Method for linear ODE</u>:- The second type of solution technique we consider for ODE-BVP is based on replacing the derivatives in the differential equation by finite-difference approximations. The interval of interest, $[a,b]$ is divided into n subintervals by specifying evenly spaced values of the independent variable, x_0, x_1, x_2 --- x_n with $x_0 = a$ and $x_n = b$. Each subinterval is of length $h = x_{i+1} - x_i$. The approximate solution at x_i is denoted y_i.

Therefore by central difference formula

$$y''(x_i) \simeq \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

Ex)-ii.) Apply finite difference method to solve the given problem: $\frac{d^2y}{dx^2} = y + x(x-4)$, $0 \le x \le 4$

with $y(0)=0$, $y(4)=0$ & $h=1$

Soln)- The finite difference method will find an approximate solution at the points $x_1=1$, $x_2=2$, & $x_3=3$ using the central difference formula for second derivative, we find that the differential equation becomes the system

$$y''(x_i) \approx \frac{y_{i+1} - 2y_i - y_{i-1}}{h^2} = y_i + x_i(x_i-4), \quad i = 1, 2, 3$$

Here $h=1$ Substitute the value of x_1, x_2 & x_3

Therefore

$$\frac{y_2 - 2y_1 + y_0}{(1)^2} = y_1 + x_1(x_1-4)$$

$$\frac{y_3 - 2y_2 + y_1}{(1)^2} = y_2 + x_2(x_2-4)$$

$$\frac{y_4 - 2y_3 + y_2}{(1)^2} = y_3 + x_3(x_3-4)$$

$$y_2 - 2y_1 + 0 = y_1 + 1(1-4)$$
$$y_3 - 2y_2 + y_1 = y_2 + 2(2-4)$$
$$0 - 2y_3 + y_2 = y_3 + 3(3-4)$$

On simplyfying

$$-3y_1 + y_2 = -3$$
$$y_1 - 3y_2 + y_3 = -4$$
$$y_2 - 3y_3 = -3$$

Solving the above equation, we get

$$y_1 = \frac{13}{7}, \quad y_2 = \frac{18}{7}, \quad y_3 = \frac{13}{7}$$

Ex:- (6.1): Using the finite difference method, find $y(0.25)$, $y(0.5)$, $y(0.75)$ satisfying the differential equation $\frac{d^2y}{dx^2} - y = x$ with condition $y(0)=0$, $y(1)=1$.

Soln:- Given that $y(0)=0$, $y(1)=y_4=1$, $h=0.25$

$y(0.25)=y_1$, $y(0.5)=y_2$, $y(0.75)=y_3$

Now using central difference formula for second derivative

$$y''(x_i) \approx \frac{y_{i+1}-2y_i-y_{i-1}}{h^2} = x_i+y_i \quad , i=1,2,3$$

Substitute the value of x_1, x_2 & x_3

$$\frac{y_2-2y_1-y_0}{(0.25)^2} = x_1+y_1$$

$$\frac{y_3-2y_2-y_1}{(0.25)^2} = x_2+y_2$$

$$\frac{y_4-2y_3-y_2}{(0.25)^2} = x_3+y_3$$

$$\frac{y_2-2y_1-0}{0.0625} = 0.25+y_1$$

$$\frac{y_3-2y_2-y_1}{0.0625} = 0.50+y_2$$

$$\frac{1-2y_3-y_2}{0.0625} = 0.75+y_3$$ On simplifying we get

$$-2.0625y_1+y_2 = 0.015625$$

$$y_1-2.0625y_2+y_3 = 0.03125$$

$$y_2-2.0625y_3 = -0.953125$$

Solving the above equation, we get

$y_1=0.180189$, $y_2=0.387266$, $y_3=0.649886$

Hence $y(0.25)=0.180189$, $y(0.5)=0.387266$, $y(0.75)=0.649886$

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Floating Point Arithmetic :- Floating point representation :- Floating point representation is also known as real point representation. It uses the number with fractional parts as operands. Example: If we consider a number 2.366, in this 2 is fixed point and 0.366 is the Fractional part known as Floating point.

Here $F = (\beta, k, m, M)$ = Floating point system

where β = base, k = no. of digits in basic expansion, m = minimum exponent, M = maximum exponent.

× Elements of $F(\beta, k, m, M)$ can also be expressed as $x = \pm(0.d_1 d_2 - d_n)_\beta \beta^e$ where $d_1, d_2, - d_n$ are all digits in base β and all d's lie between 0 & β.

Note: A floating point $x = \pm(0.d_1 d_2 - d_n)_\beta \beta^e$ is said to be normalized if $d_1 \neq 0$ or else $d_1 = d_2 = - - = d_n = 0$.

Arithmetic Operations :- (1.) Addition of Normalized Floating Point :-

For adding two normalized floating points, we first have to make their exponents equal by shifting the mantissa.

Ex1. (1.) Add 0.7642 E4 & 0.4253 E6

Soln: 0.7642 E4 mean 0.7642×10^4

& 0.4253 E6 mean 0.4253×10^6

The exponent of a number with the smallest exponent is increased by 2 so that 0.7642 becomes 0.0076 E6

Then 0·7642 E4 + 0·4253 E6 = 0·0076 E6 + 0·4253 E6

= 0·4329 E6.

Ex:(2) Represent $33{\cdot}58 \times 10^6$ in normalized floating point mode.

Soln: $33{\cdot}58 \times 10^6$, after normalization It can be written as $0{\cdot}3358 \times 10^8$ or 0·3358 E8.

Ex:(3.) Add 0·5464 E3 + 0·5445 E8

Soln:- 0·5464 E3 means $0{\cdot}5464 \times 10^3$

0·5445 E8 means $0{\cdot}5445 \times 10^8$

The exponent of a number with smallest exponent is increased by 5 so that 0·5464 becomes 0·000005464 E8

Then 0·5464 E3 + 0·5445 E8 = 0·0000 E8 + 0·5445 E8

= 0·5445 E8.

(2.) Subtraction of Normalized Floating Point:- This operation is performed by adding negative normalized floating points.

Ex:-(1.): Subtract 0·4673 E-4 ~~from~~ 0·8542 E-5

Soln:- The smallest exponent is E-5 so we increase the exponent of 0·8542 E-5 by 1 and It becomes 0·0854 E-4

So 0·4673 E-4 − 0·0854 E-4 = 0·3819 E-4

Ex:-(2.): (0·4932 E-8) − (0·8362 E-9) Subtract

Soln: The smallest exponent is E-9 so we Increase the exponent 0·8362 E-9 by 1 and It becomes 0·0836 E-8

So 0·4932 E-8 − 0·0836 E-8 = 0·4096 E-8

(3.) Multiplication of Normalized Floating Point :- In order to multiply two floating points, we multiply their mantissa and add their exponents. The mantissa is only four digit of the resulting mantissa which are retained by dropping the rest of the digits.

Ex:-(1.): Multiply 0.5634 E11 × 0.1532 E-14

Soln: 0.5634 × 0.1532 = 0.08631288 & E11 × E-14 = E-3

Therefore 0.5634 E11 × 0.1532 E-14 = 0.08631288 E-3

Now the leading digit of mantissa should be nonzero therefore

0.08631288 E-3 becomes 0.8631288 E-4 or 0.8631 E-4

Ex:-(2.): Multiply 0.1222 E10 × 0.2143 E15

Soln: 0.1222 × 0.2143 = 0.02618746 & E10 × E15 = E25

Therefore, 0.1222 E10 × 0.2143 E15 = 0.02618746 E25

= 0.2618 E24.

(4.) Divison of Normalized Floating Point :- In this operation the mantissa of numerator is divided by the mantissa of the denominator and the exponent of the denominator is subtracted from the exponent of the numerator. The quotient mantissa obtained by is normalized by retaining 4 digits and the exponent is suitably adjusted.

Ex:-(1.):- Divide 0.2000 E5 by 0.8883 E3

Soln:- $\frac{0.2000}{0.8883} = 0.2251$ & $\frac{E5}{E3} = E2$

Therefore $\frac{0.2000\,E5}{0.8883\,E3} = 0.2251\,E2$.

Ex1-(2.7): Divide 0.8889 E1 ÷ 0.2000 E1

Soln:- $\frac{0.8889}{0.2000} = 4.4445$ & $\frac{E1}{E1} = E0$

Therefore $\frac{0.8889E1}{0.2000E1} = 4.4445 = 0.4444E1$.

Significant Digits:- The digits 1, 2, 3, —— 9 that are used to express a number are called significant digits or significant figures, '0' is also a significant figure except when it is used to fix the decimal point or fill the places of unknowns or discarded digits.

Example:- The number 0.00123 has only three significant digits viz. 1, 2 & 3. The number 0.66753, 3.1416, 3.0687 each contains five significant digits.

Errors:- In any numerical computation, there are several types of error.

(1.) Inherent Error: Errors which are already present in the problem even before its solution are called inherent error.

(2.) Rounding off Error:- Errors which arises in the process of rounding off the numbers during computations.

(3.) Absolute Error:- Absolute error is the numerical difference between the true value of a quantity and its approximate value. If Y is the true value of a quantity and Y' be its approximate value, then |Y - Y'| is the absolute error denoted by $E_a = |Y - Y'|$

Example:- let $Y = 1.253$ & $Y' = 1.25$

$E_a = |1.253 - 1.25| = 0.003$.

4. Relative error :- The relative error is the absolute error divided by the magnitude of the exact value.

It is given by $E_r = \left|\frac{y-y'}{y}\right|$

Ex:- Let $y = 1.253$ & $y' = 1.25$

$$E_r = \left|\frac{y-y'}{y}\right| = \left|\frac{0.003}{1.253}\right| = 0.0023$$

5. Percentage Error :- The percentage error is the relative error expressed in terms of per 100 and is given by

$$E_p = 100 \times E_r = 100\left|\frac{y-y'}{y}\right|$$

Ex: $E_p = 100\left|\frac{0.003}{1.253}\right| = 0.23$

Truncation Errors or Local Truncation Error :- Errors which are caused by using approximate formulae in computation or replacing an infinite process by a finite process are called truncation errors.

For example :- Consider an exponential series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \infty = X \text{ (say)}$$

It is replaced by $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = X'$ (say)

Then truncation error $= X - X'$

Global Truncation Error :- It is used to improve the results. i.e. Global error of a multi-step method is the difference : $E_n(x) = y(x) - y(x_n)$ where $x = x_n$ is fixed and n is a variable. Subdivide the interval into n equal sub-interval and apply the method in each region.

For example :- If we take general trapezoidal rule is

$\int_a^b f(x)\,dx \approx \frac{h}{2}[f(a) + 2f(x_1) + \cdots 2f(x_{n-1}) + f(b)]$. If f is sufficiently smooth, the global error can be represented as $-\frac{(b-a)h^2}{12} f''(\eta)$ for some $a \le \eta \le b$. Thus the global error for the trapezoidal rule is proportional to h^2 & the method is of order h^2.

Ex:-(1.) Round off the numbers 665250 and 27.46235 to four significant digits and compute absolute error, relative error & percentage error.

Soln:
(i) 665250 is rounded off to four significant digits = 665200

Here $x = 665250$, $x' = 665200$

Absolute error $= |x - x'| = |665250 - 665200| = 50$

Relative error $= \left|\frac{x-x'}{x}\right| = \frac{50}{665250} = 7.52 \times 10^{-5}$

Percentage error $= \left|\frac{x-x'}{x}\right| \times 100 = 7.52 \times 10^{-5} \times 100 = 7.52 \times 10^{-3}$

(ii) 27.46235 is rounded off to four significant digit = 27.46

Here $x = 27.46235$, $x' = 27.46$

Absolute error $= |x - x'| = |27.46235 - 27.46| = 0.00235$

Relative error $= \left|\frac{x-x'}{x}\right| = \frac{0.00235}{27.46235} = 8.56 \times 10^{-5}$

Percentage error $= \left|\frac{x-x'}{x}\right| \times 100 = 8.56 \times 10^{-5} = 8.56 \times 10^{-3}$

Ex:-(2.):- An approximate value of π is given by 3.1428571 and its true value is 3.1415926. Find absolute & relative errors.

Soln:- Here $x = 3.1415926$, $x' = 3.1428571$

Absolute error $= |x - x'| = |3.1415926 - 3.1428571| = 0.0012645$

Relative error $= \left|\frac{x-x'}{x}\right| = \frac{0.0012645}{3.1415926} = 4.02 \times 10^{-4}$.