Power Series

Find Radius of Conv. & Interval of Conv.

(Q) $\sum \frac{(-1)^n (x-1)^n}{2^n (3n-1)}$, here $a_n = \frac{(-1)^n}{2^n (3n-1)}$

Sol^n The given power series is about the point $x=1$, we have :-

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{2^{n+1}(3n+2)}{2^n(3n-1)}$$

$$= 2.$$

$$\left[a_{n+1} = \frac{(-1)^{n+1}}{2^{n+1}(3n+2)} \right.$$

Thus $R = 2$

The domain of convergence of given power series is :-

$$-2+1 < x < 2+1$$

or $-1 < x < 3$.

For $x = 3$, given power series is: $\sum \frac{(-1)^n}{(3n-1)}$, which is convergent by Leibnitz's Test. Hence domain of convergence of given power series is:- $-1 < x \leq 3$.

$$\left\{ \begin{array}{l} \sum \frac{(-1)^n \cdot 2^n}{2^n (3n-1)} = \sum \frac{(-1)^n}{3n-1} \\ \text{Let } u_n = \frac{1}{3n-1}, \ \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{3n-1} = 0 \\ \text{and } u_{n+1} < u_n \ \forall n \end{array} \right.$$

For $x = -1$, given power series is

$$\sum \frac{(-1)^n (-2)^n}{2^n (3n-1)} = \sum \frac{(-1)^{2n} \cdot 2^n}{2^n \cdot (3n-1)}$$

$$= \sum \frac{(-1)^{2n}}{3n-1} = \sum \frac{1}{3n-1}$$

Let $u_n = \frac{1}{3n-1}$, Let $v_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n}{3n-1} = \lim_{n\to\infty} \frac{n}{n(3-1/n)}$$

$$= \frac{1}{3} \neq 0 \text{ \& finite}$$

Hence $\sum u_n$ & $\sum v_n$ cgs or dvgs together

Since $\sum v_n$ diverges

$\therefore$ $\sum u_n$ diverges

Hence, the exact interval of convergence is $]-1, 3]$.

Determine the interval of conv. of the Pow

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x-1)^n}{n}$$

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{n+1}{n}\right| = 1$$

$\therefore \quad R = 1$

Interval of conv.

$$-1+1 < x < 1+1$$

$$\Rightarrow 0 < x < 2$$

$$\Rightarrow x \in \,]0, 2[$$

For $x = 2$

The series $\sum \frac{(-1)^{n+1}}{n}$ is cgt by

Leibnitz's Test

$\therefore$ interval is

$$]0, 2]$$

Cauchy Hadamard Thm: [power] of convergence of the Power Series

If R is the radius of convergence

then the series is absolutely cgt if $|x|<R$

and dgt if $|x|>R$.

Case 1. $R=\infty$

(i) Suppose $\underset{n}{\mathrm{lt}}\sup|a_n|^{1/n}=0$.

Now $\underset{n}{\mathrm{lt}}\sup|a_n x^n|^{1/n} = \underset{n}{\mathrm{lt}}\sup\left(|x|\,|a_n|^{1/n}\right)$

$$= |x|\,\underset{n}{\mathrm{lt}}\sup|a_n|^{1/n} = |x|.0$$

Thus there exists a positive integer N such

$$|a_n x^n|^{1/n} < 1/2 \quad \forall\ n\geq N$$

ie $|a_n x^n| < \left(\frac{1}{2}\right)^n \quad \forall\ n\geq N$

geometric series $\sum\left(\frac{1}{2}\right)^n$ is cg[t]

Comparison test the series $\sum|a_n x^n|$ is

hence $\sum a_n x^n$ is absolutely cgt

every x

$\sum$ ... absolutely ... for every x.

Case 2 : $R = 0$

Suppose $\lim_{n} \sup |a_n|^{1/n} = \infty$.

For any $x \neq 0$,

$$\lim_{n} \sup |a_n x^n|^{1/n} = |x| \cdot \lim_{n} \sup |a_n|^{1/n} = \infty.$$

Thus $|a_n x^n| > 1$ for infinitely many values of n

Hence $\lim_{n\to\infty} a_n x^n \neq 0$, for every $x \neq 0$.

Thus the series $\sum a_n x^n$ diverges for every x

Case III

(iii) Suppose $0 < R < \infty$ and $\lim\sup |a_n|^{1/n} = \frac{1}{R}$

For $0 < |x| < R$, $\exists$ $0 < k < 1$ such that $|x| < kR$. ---- (1)

Now $\lim\sup_n |a_n x^n|^{1/n} = \lim\sup_n (|x| |a_n|^{1/n}) = |x| \lim\sup_n |a_n|^{1/n}$

$= \frac{|x|}{R} < k$ by (1)

Thus $\exists$ a positive integer N s.t.

$$|a_n x^n|^{1/n} < k \quad \forall n \geq N$$

i.e $|a_n x^n| < k^n \quad \forall n \geq N$

Since $0 < k < 1$, the geometric series $\sum k^n$ is cgt.

Hence by comparison test the series

$$\sum a_n x^n$$

is absolutely cgt.

Let $|x| > R$. Now

$$\lim\sup_n |a_n x^n|^{1/n} = |x| \lim\sup_n |a_n|^{1/n} = \frac{|x|}{R} > 1$$

$\therefore$ $|a_n x^n| > 1$ for infinitely many value of

Thus seq $\{a_n x^n\}$ does not cgs to 0. Hence

the series $\sum a_n x^n$ is dgt for $|x| > R$

Q: Find the radius of convergence and interval of convergence of the series $\sum \frac{x^n}{n}$ and $\sum \frac{x^n}{n^2}$.

Here $a_n = \frac{1}{n}$

By def

$$R = \frac{1}{\underset{n}{lt} \sup |a_n|^{1/n}} = \frac{1}{lt \sup \left(\frac{1}{n}\right)^{1/n}}$$

$$= \frac{1}{1} = 1$$

$\left[\because \underset{n \to \infty}{lt} (n)^{1/n} = 1\right]$

$\left\{\begin{array}{l}\text{If } lt \text{ exists} \\ \text{Then } lt = \overline{lt} = \underline{lt} \\ \text{i.e. } lt = lt \sup = lt \inf\end{array}\right.$

$\therefore$ $R = 1$

$\therefore$ Radius of conv. is 1

Interval of " is $]-1, 1[$

By Cauchys Hadamard Thm.

$\sum \frac{x^n}{n}$ is abs. cgt for $|x| < 1$

& div. for $|x| > 1$.

For $x = 1$, $\sum \frac{1}{n}$ diverges.

For $x = -1$, $\sum \frac{(-1)^n}{n}$ is convergent (by Leibnitz test for alternating series).

Find radius and interval of conv. when a_n is given by $a_n = \frac{1}{n^n}$

$$\underset{n}{lt} \sup |a_n|^{1/n} = \underset{n}{lt} \sup \left|\frac{1}{n^n}\right|^{1/n}$$

$$= \underset{n}{lt} \sup \frac{1}{n} = 0$$

$\therefore$ Radius of conv $= R = \infty$

Interval of conv. $=]-\infty, \infty[$