A set of ordered pairs

$$P = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$$

of subintervals and corresponding tags is called a **tagged** partition of I

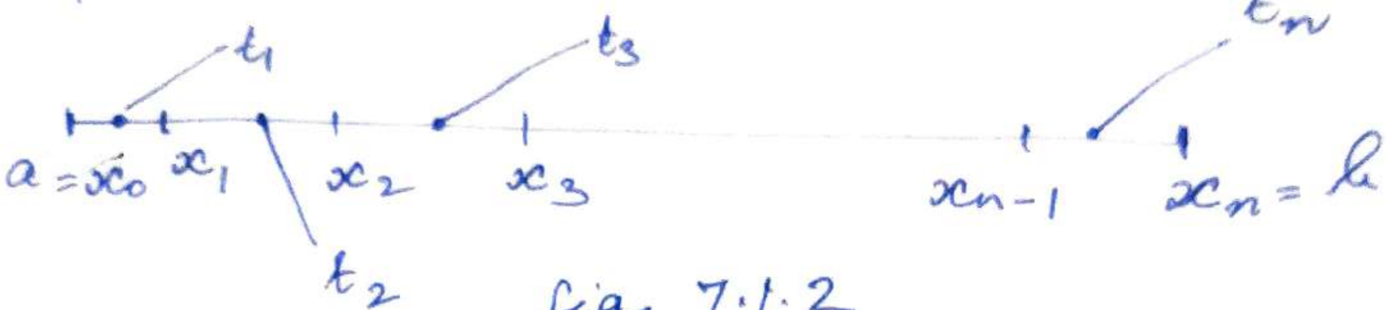
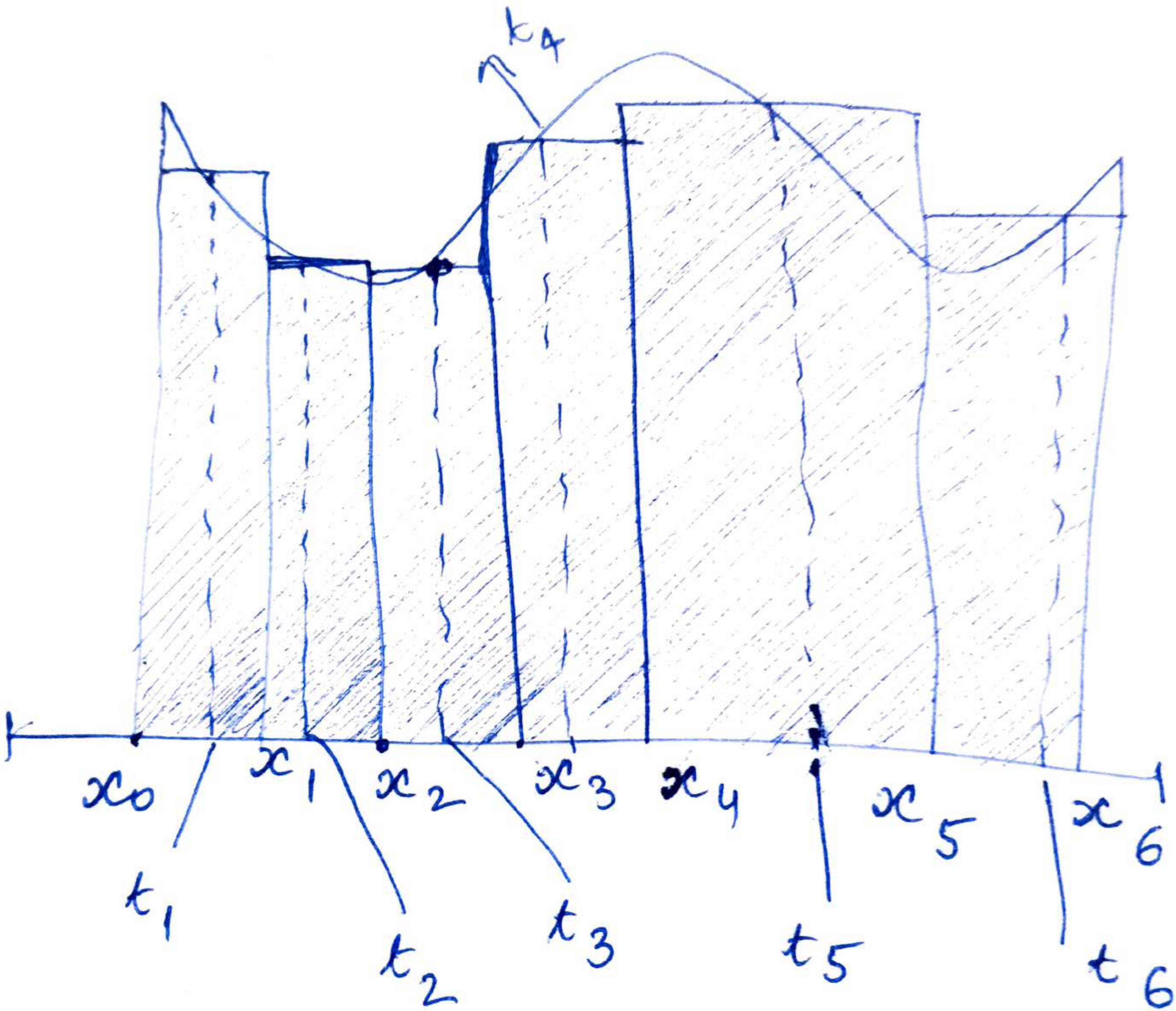


fig. 7.1.2

(The dot over the partion P indicates that a tag has been choosen for each subinterval) ation :



Definition

Let f be a bounded fn on $[a,b]$.

Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$.

~~Then~~ Corresponding to a partition P of $[a,b]$ let us choose points $t_1, t_2, \ldots, t_n$ s.t.

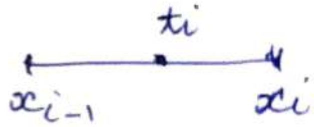
$$x_{i-1} \le t_i \le x_i \quad (i = 1, 2 \ldots n)$$

and let us consider the sum

$$S(P,f) = \sum_{i=1}^{n} f(t_i)\,\Delta x_i \qquad (\Delta x_i = x_i - x_{i-1})$$

The sum $S(P,f)$ is called a Riemann Sum of f over $[a,b]$ relative to P.

Note : t_i are arbitrary, t_i can be any pt. whatsoever of Δx_i



→ A bounded fn $f : [a,b] \to \mathbb{R}$ is said to be Riemann integrable on $[a,b]$ if $\exists$ a real no. A s.t. for every $\epsilon > 0$ $\exists$ $\delta > 0$ s.t.

$$\mu(P) < \delta \Rightarrow |S(P,f) - A| < \epsilon$$

A is called Riemann integral of f over $[a,b]$ and we write

$$A = \int_a^b f = \int_a^b f(x)\,dx$$

and $f \in \mathcal{R}[a,b]$

Thm : A bdd fn f on $[a,b]$ is Riemann Integrable iff it is [Darboux] integrable, in which case the values of integrals agree.

(3)

Show that $\int_1^2 f\,dx = \frac{11}{2}$, where $f(x) = 3x+1$

Sol : Let $P = \{1 = x_0, x_1, x_2 \ldots, x_n = 2\}$ be a partition which divides $[1,2]$ into n equal sub-intervals, each of length $\frac{2-1}{n} = \frac{1}{n}$, so that

$$\mu(P) = \frac{1}{n} \to 0 \quad \text{as } n \to \infty$$

$$x_i = 1 + \frac{i}{n}, \quad i = 1, 2 \ldots n$$

$$\Delta x_i = \frac{1}{n}, \quad i = 1, 2 \ldots n$$

$$\sum_{i=1}^{n} \Delta x_i = n \cdot \frac{1}{n} = 1$$

Let $t_i = x_i$ when $i = 1, 2 \ldots n$

$$\therefore S(P,f) = \sum_{i=1}^{n} f(t_i)\,\Delta x_i$$

$$= \sum_{i=1}^{n} f(x_i)\,\Delta x_i$$

$$= \sum_{i=1}^{n} (3x_i + 1)\,\Delta x_i$$

$$= \sum_{i=1}^{n} \left\{3\left(1 + \frac{i}{n}\right) + 1\right\} \Delta x_i$$

$$= 4\sum_{i=1}^{n} \Delta x_i + \frac{3}{n^2} \sum i$$

$$= 4 + \frac{3}{n^2} \frac{n(n+1)}{2} = \frac{11}{2} + \frac{3}{2n}$$

Taking limits $\mu(P) \to 0$

$$\lim_{\mu(P) \to 0} S(P,f) = \frac{11}{2} \qquad \left[\begin{array}{c}\mu(P) = \|P\| \\ \text{norm } P\end{array}\right]$$

Since the lt. exists, the fn is integrable &

$$\boxed{\int_1^2 f\,dx = \lim S(P,f) = \frac{11}{2}}$$

Thm : Every continuous fn is integrable. (4)

Pf : Let f be continuous fn in $[a,b]$ (closed & bdd interval)

$\therefore$ f is uniformly continuous on $[a,b]$

$\therefore$ given $\epsilon > 0$, $\exists\ \delta > 0$, s.t. $\forall$ pair of points $p, q \in [a,b]$ with $|p-q| < \delta$

$$\Rightarrow \quad |f(p) - f(q)| < \frac{\epsilon}{b-a} \quad \text{—— (1)}$$

(by def. of Unif. conti)

Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be a partition of $[a,b]$ with norm $\mu(P) < \delta$.

Let $M_i = \sup\limits_{[x_{i-1}, x_i]} f(x)$

$m_i = \inf\limits_{[x_{i-1}, x_i]} f(x)$

As f is continuous on $[a,b]$, is continuous over every subinterval of $[a,b]$.

$\therefore$ the supremum & infimum over each sub interval is attained.

$\therefore\ \exists\ p_i$ & $q_i \in [x_{i-1}, x_i]$ s.t.

$f(p_i) = M_i$ & $f(q_i) = m_i$ &

$|x_i - x_{i-1}| < \delta$

$\therefore\ |p_i - q_i| < |x_i - x_{i-1}| < \delta$

$\Rightarrow\ |f(p_i) - f(q_i)| < \frac{\epsilon}{b-a}$. (using (1))

Now $U(P,f) - L(P,f) = \sum\limits_{i=1}^{n} (M_i - m_i)\, \Delta x_i$

$= \sum\limits_{i=1}^{n} \left(f(p_i) - f(q_i)\right) \Delta x_i$ $\quad \because f(p_i) - f(q_i) < \frac{\epsilon}{b-a}$

$= \sum\limits_{i=1}^{n} \frac{\epsilon}{b-a}\, \Delta x_i$

$= \frac{\epsilon}{b-a} \sum\limits_{i=1}^{n} \Delta x_i = \frac{\epsilon}{(b-a)} \cdot (b-a) = \epsilon$

$\therefore \; U(P,f) - L(P,f) < \epsilon \quad \forall$ partition P with $\mu(P) < \delta$.

$\therefore$ f is integrable.

: If f is monotonic on $[a,b]$, then $f \in \mathbb{R}[a,b]$.

: Suppose f is monotonic increasing on $[a,b]$

Then $f(a) \le f(x) \le f(b) \quad \forall \; x \in [a,b]$.

$\therefore$ f is bdd on $[a,b]$

& $\inf f = f(a)$, $\sup f = f(b)$

Let $\epsilon > 0$ be given

Let $P = \{a = x_0, x_1 \; . \; . \; . \; x_n = b\}$ be a partition of $[a,b]$ with mesh$(P) < \dfrac{\epsilon}{[f(b) - f(a) + 1]}$

If m_i & M_i be the inf & sup of f on $I_i = [x_{i-1}, x_i]$

Then $m_{i-1} = f(x_{i-1})$ &

$M_i = f(x_i)$ $\quad$ ($\because$ f is m.i on $[a,b]$)

because f is m.i. on $[a,b]$.

Hence

$$U(P,f) - L(P,f) = \sum_{i=1}^{n} (M_i - m_i)\,\Delta x_i$$

$$= \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]\,\Delta x_i$$

$$\le \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \cdot \frac{\epsilon}{f(b) - f(a) + 1}$$

$$= \frac{\epsilon}{f(b) - f(a) + 1} \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]$$

$$= \frac{\epsilon}{f(b) - f(a) + 1} \Big[f(x_1) - f(x_0) + f(x_2) - f(x_1) + \; .... \; + f(x_n) - f(x_{n-1}) \Big]$$

$$= \frac{\epsilon}{f(b) - f(a) + 1} [f(b) - f(a)] \qquad \left[\because \begin{array}{l} f(x_0) = f(a) \\ f(x_n) = f(b) \end{array} \right.$$

$< \epsilon$

$\therefore$ f is integrable.

<u>Note</u> : $f(b) - f(a) + 1$ has been taken to cover the case when $f(a) = f(b)$.

$\because$ f is monotonic decreasing on $[a, b]$

$\therefore$ $-f$ is monotonic increasing on $[a, b]$

So $-f \in R[a, b]$

$$\therefore \underline{\int_a^b} -f(x)\,dx = \overline{\int_a^b} -f(x)\,dx$$

$$\text{i.e. } -\underline{\int_a^b} f(x)\,dx = -\overline{\int_a^b} f(x)\,dx$$

$$\text{i.e. } \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$

$\therefore$ f is R-integrable on $[a, b]$.