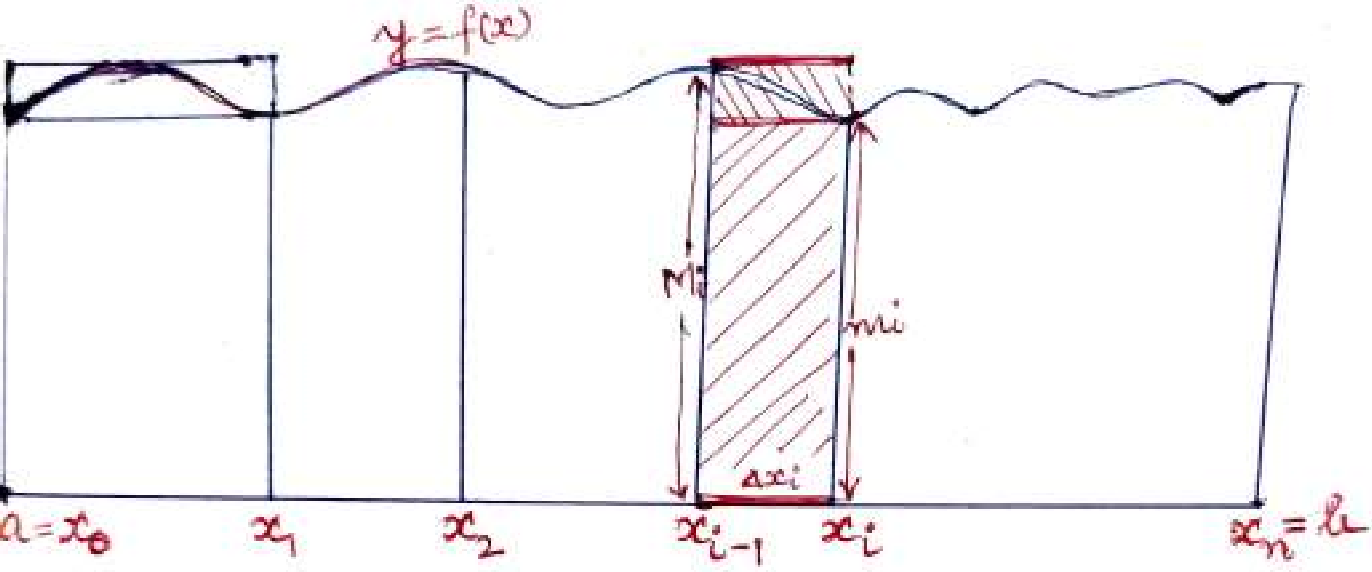
Riemann Integration



For

A bounded function f over [a, b]

if upper integral $\overline{\int_a^b} f$ = Lower integral $\underline{\int_a^b} f$,

then f is said to be Riemann Integral over [a, b] and is denoted by $\int_a^b f$.

Q.1. Define Riemann integral of a bounded function defined on a closed and bounded interval $[a, b]$.

Ans: Defn:-

Let $f : [a, b] \longrightarrow \mathbb{R}$ be bounded function.

Let $P = \{a = x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_{n-1} \leq x_n = b\}$ be a partition of $[a, b]$.

Then f is bdd in each of the sub-intervals $[x_{i-1}, x_i]$, corresponding to the partition P.

Let $\Delta x_i = x_i - x_{i-1}$ for $i = 1, 2, 3, \ldots$

And let $M_i = \sup\limits_{x \in \Delta x_i} f(x)$ and $m_i = \inf\limits_{x \in \Delta x_i} f(x)$.

Then,

upper Sum of f over P $= U(P, f) = \sum\limits_{i=1}^{n} M_i \Delta x_i = M_1 \Delta x_1 + \ldots + M_n \Delta x_n$,

Lower Sum of f over P $= L(P, f) = \sum\limits_{i=1}^{n} m_i \Delta x_i = m_1 \Delta x_1 + \ldots + m_n \Delta x_n$

are called upper sum and lower sum ~~respectively~~ of ~~the~~ f respectively corresponding to the partition P.

Now, the upper integral of f over $[a, b]$, denoted as $\overline{\int_a^b} f\,dx$ is defined as

$$\overline{\int_a^b} f\,dx = \inf\{U(P, f) : P \text{ is a partition of } [a, b]\}.$$

And the lower integral of f over $[a, b]$, denoted as $\underline{\int_a^b} f\,dx$, is defined as

$$\underline{\int_a^b} f\,dx = \sup\{L(P, f) : P \text{ is a partition of } [a, b]\}.$$

the two integrals are equal, the integral is

Riemann integral.

and so

$$\overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx = \int_a^b f\,dx, \quad f \in R.$$

The upper integral $U(f)$ of f over $[a,b]$ is
..ly $U(f) = \inf \{U(f,P) : P$ is a partition ...
lower integral is

$L(f) = \sup \{L(f,P) : P$ is a partition

We say that f is integrable on $[a,b]$ pr...

$L(f) = U(f)$

In this case we write $\int_a^b f$ or $\int_a^b f(x)dx$
common value:

$$\int_a^b f = \int_a^b f(x)dx = L(f) = U(f)$$

...rtition for $[a,b] = [0,1]$, $a=0$

$$P = \{0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < 1\}$$

Show that x^2 is integ. on any interva...

Consider the partition of $[0,b]$

$$P = \{0, b/n, \frac{2b}{n}, \ldots, \frac{(i-1)b}{n}, \frac{ib}{n}, \ldots$$

$\int_a f = \int_a f(x)dx =$

…tion for $[a,b] = [0,1]$, $a=0$

$P = \{0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < 1\}$

Show that x^2 is integ. on any interval $[0,b]$ …

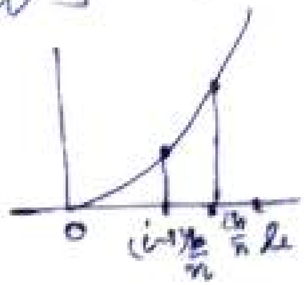
…nsider the partition of $[0,b]$

$P = \{0, b/n, \frac{2b}{n}, \dots, \frac{(i-1)b}{n}, \frac{ib}{n}, \dots \frac{nb}{n} = b\}$

Then $\Delta x_i = b/n$

bounds of $f(x) = x^2$ in $[(i-1)\frac{b}{n}, \frac{ib}{n}]$ are

$M_i = \left(\frac{ib}{n}\right)^2$, $m_i = \left[\frac{(i-1)b}{n}\right]^2$



$$(f, P) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \left(\frac{ib}{n}\right)^2 \frac{b}{n}$$

$$= \frac{b^3}{n^3} \sum_{i=1}^{n} i^2 = \frac{b^3}{n^3}(1^2 + 2^2 + \cdots + n^2)$$

$$= \frac{b^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{b^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right.$$

as $n \to \infty$

$$U(f,P) \to \frac{b^3}{3}$$

$$\therefore \lim_{n\to\infty} U(f,P) = \frac{b^3}{3}.$$

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta t_i$$

$$= \sum_{i=1}^{n} \frac{(i-1)^2}{n^2} b^2 \cdot \frac{b}{n}$$

$$= \frac{b^3}{n^3}\left(1^2 + 2^2 + \cdots + (n-1)^2\right)$$

$$= \frac{b^3}{n^3} \frac{(n-1)(n)(2n-1)}{6}$$

$$= \frac{b^3}{6}\left(1-\frac{1}{n}\right)\left(\frac{n}{n}\right)\left(2-\frac{1}{n}\right)$$

$$\therefore \lim_{n\to\infty} L(f,P) = \frac{b^3}{3} \left(= \frac{1\cdot 1\cdot 2}{6} \times b^3\right)$$

$$\therefore \boxed{\int_a^b f\,dx = \int_a^b x^2\,dx = \frac{b^3}{3}}$$

Ex: 2. If $f(x) = \begin{cases} 0 & \text{when } x \text{ is rat} \\ 1 & \text{'' } x \text{ is} \end{cases}$

integrable in any

$$= \frac{b^3}{n^3} \frac{(n-1)(n)(2n-1)}{6}$$

$$= \frac{b^3}{6} \left(1-\frac{1}{n}\right)\left(\frac{n}{n}\right)\left(2-\frac{1}{n}\right)$$

$$\therefore \lim_{n\to\infty} L(f,P) = \frac{b^3}{3} \left(= \frac{1.1.2}{6} \times b^3 \right)$$

$$\therefore \boxed{\int_a^b f\,dx = \int_a^b x^2\,dx = \frac{b^3}{3}}$$

Ex: 2. If $f(x) = \begin{cases} 0 & \text{when } x \text{ is rational} \\ 1 & \text{when } x \text{ is irrational} \end{cases}$

Show that f is not integrable on any interval.

Let $P = \{a = t_0 < t_1 < \cdots\cdots t_n = b\}$ be any partition of an inter. $[a,b]$. $(a \neq b)$

Then $U(f,P) = \sum_{i=1}^{n} M_i \Delta t_i)$ $\quad M_i = 1$, $m_i = 0$ $(i = 1, 2 \ldots n)$

$$= \sum_{i=1}^{n} 1 \cdot \Delta t_i \qquad \Delta t_i = (b-a)$$

$$= (b-a)$$

$L(f,P) = \sum m_i \Delta t_i = 0 \cdot \Delta t_i = 0$.

$\therefore U(f) = (b-a)$, $L(f) = 0$ f is not inte

Definition :

We define Norm $P = \max\{\Delta x_i : i = 1, 2 \ldots n\}$ and denote it by $\|P\|$.

Refinement :

If P and Q are two partitions of $[a, b]$ s.t

$$P \subseteq Q,$$

Then Q is called refinement of P.