← Cauchy - Hadamard Theorem:

If R is the radius of convergence of the P.S. $\Sigma a_n x^n$, then the series is absolutely convergent if $|x| < R$
& divergent if $|x| > R$.

Note: It will be noted that the Cauchy-Hadamard Thm makes no statement as to whether the P.S. converges when $|x| = R$.

Assume a fn f is represented by the P.S
$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$
in the interval of convergence $(a-h, a+h)$

Then we have:

(a) The differentiated series $\sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$ also has radius of convergence h

(b) The derivative $f'(x)$ exists for each x in the interval of convergence and is given by
$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$$

If a p.s. converges at an end pt of its interval of convergence, then the p.s. is uniformly convergent in the interval which includes this end pt.

k : (1) If a p.s. with interval of conv. $]-R, R[$ converges at both the end pts $-R$ and R, then the p.s. is uniformly convergent in $[-R, R]$.

If a p.s. converges at the end pt. $x = R$ of the interval of convergence $]-R, R[$ then it is uniformly convergent in]

or $[-R+\epsilon, R]$, where $\epsilon > 0$.
In particular, the series is uniformly convergent in $[0, R]$.

Thm: Let $\sum a_n x^n$ be a p.s. with radius of convergence R and let

$$f(x) = \sum a_n x^n \quad , \quad -R < x < R$$

If the series $\sum a_n R^n$ converges, then prove that

$$\lim_{x \to R-0} f(x) = \sum a_n R^n.$$

: The case $R = 1$.

Let $\sum_{0}^{\infty} a_n x^n$ be a p.s. with unit radius of convergence and let

$$f(x) = \sum_{0}^{\infty} a_n x^n \quad , \quad -1 < x < 1$$

If the series $\sum a_n$ converges then prove that

$$\lim_{x \to 1-0} f(x) = \sum_{0}^{\infty} a_n$$

Show that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad -1 < x < 1 \quad (1)$$

We know

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots, \quad -1 < x < 1$$

It's radius of convergence is $R = 1$.

Thus the p series is absolutely convergent for $|x| < 1$ and uniformly convergent for $[-k, k]$, $|k| < 1$

The integrated series is also convergent absolutely in $]-1, 1[$ and uniformly in $[-k, k]$, $|k| < 1$

Integrating (1) we get

$$\log(1+x) = c + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$-1 < x < 1$$

Putting $x = 0$ gives $c = 0$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad -1 < x < 1$$

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$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \quad -1<x<1$$

We know

$$(1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \cdots \quad |x|<1 \qquad (1)$$

Radius of convergence is

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = 1$$

Thus the series is convergent absolutely $\forall x$ s.t. $|x|<1$ & is U.C. in $[-k,k]$, $|k|<1$.

The integrated series is also conv. absolutely in $]-1,1[$ & uniformly in $[-k,k]$, $|k|<1$.

Integrating (1) we get

$$\tan^{-1}x = \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) + c \quad |x|<1$$

where c is constant of integration.

Putting $x=0$, we get $c=0$ and so

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad |x|<1$$

Exponential fn $E(x)$ as the sum of a P.S. and its properties.

Show that the domain is the set of all real nos.

P.T. $E(x+y) = E(x)E(y) \quad \forall x, y \in \mathbb{R}$

If e denotes $E(1)$, P.T. $E(x) = e^x \quad \forall x \in \mathbb{R}$

Pf: The exponential fn $E(x)$ is defined as the sum of the P.S.

$$E(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \quad ①$$

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to \infty} \frac{(n+1)!}{n!} = \lim_{n \to \infty} (n+1) = \infty$$

Thus the radius of conv $(R) = \infty$

$\therefore$ the P.S. is everywhere convergent.

i.e. $E(x)$ is defined $\forall x \in \mathbb{R}$.

$\Rightarrow$ the domain of the exponential fn is $\mathbb{R}$.

The fn $E(x)$ is continuous & has derivatives of all orders, $\forall x \in \mathbb{R}$.

Differentiating ① we get

$E'(x) = E(x)$

$E''(x) = E(x)$

$E'''(x) = E(x) \quad \ldots \quad$ etc.

As $E^{(n)}(x) = E(x)\ \forall\ n \in N$

By Taylor's thm $\forall$ real values t & x

$E(t) = E(x) + \frac{E'(x)}{1!}(t-x) + \frac{E''(x)}{2!}(t-x)^2 + \cdots$

$E(t) = E(x) + \frac{E(x)}{1!}(t-x) + \frac{E(x)}{2!}(t-x)^2 + \cdots \}$

Replacing t by $x+y$, we get

$E(x+y) = E(x)\{1 + \frac{y}{1!} + \frac{y^2}{2!} + \cdots \}$

$= E(x)E(y)$

Thus $\boxed{E(x+y) = E(x)E(y)}\ \forall\ x, y \in \mathbb{R}$ (2)

From (1)

$E(1) = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots$

The series on the R.H.S. converges to a no. which lies b/w 2 & 3.

we denote this no. by e so that

$E(1) = e$.

(2) can be extended as

$E(x_1 + x_2 + \cdots + x_n) = E(x_1) + E(x_2) + \cdots E(x_n)$ (3)

and +ve integer n & $\forall\ x_i \in R$

Taking $x_1 = x_2 = \cdots = x_n = x$ in (3),
we obtain

$$E(nx) = \{E(x)\}^n \quad \forall\, x \in \mathbb{R},\ \forall n \in \mathbb{N} \quad \ldots (4)$$

→ For $x = 1$, we obtain

$$E(n) = e^n \quad \forall\, n \in \mathbb{N} \ldots (5)$$

$(\because E(1) = e$

→ Taking $x = m/n$ $\{m$ & n being +ve integers in (4)

$$E(m) = \{E(m/n)\}^n$$

or $$e^m = \{E(m/n)\}^n \quad \text{by (5)}$$

$$\therefore\ E(m/n) = (e^m)^{1/n} = e^{m/n}$$

Thus $\boxed{E(q) = e^q}$ $\forall$ +ve rationals q → (6)

Let x be any +ve irrational no. Then there always exists a seq. $\{x_n\}$ of +ve rational nos. s.t. $x_n \longrightarrow x$.

using (6) $E(x_n) = e^{x_n} \quad \forall\, n \in \mathbb{N}$

Since $x_n \to x$ as $n \to \infty$
and E is a cts fn. $\forall x \in \mathbb{R}$
$(\therefore E(x_n) \to E(x))$
$\therefore E(x) = e^x$ $\forall$ +ve irrationals — (7)

From (6) & (7) we conclude that
$\boxed{E(x) = e^x}$ $\forall$ +ve reals — (8)

Taking $y = -x$ in (2)
$E(0) = E(x)E(-x)$
or $1 = E(x)E(-x)$ $(x \geq 0)$ — (9)
$\because E(0) = 1$ by (1)

or $E(-x) = \frac{1}{E(x)} = \frac{1}{e^x}$
$= e^{-x}$ using (8)

Hence $\boxed{E(x) = e^x}$ holds for all real x.

Monotonicity :

By def $E(x) > 0$ $\forall x > 0$
so that from (9) it follows that
$E(-x) > 0$ $\forall x > 0$
Hence $E(x) > 0$ $\forall$ real x
Again by def, for real x,

(*) $E(x) \longrightarrow +\infty$, as $x \longrightarrow +\infty$

Hence (9) shows that

$E(x) \longrightarrow 0$ as $x \longrightarrow -\infty$

Also by def

$0 < x_1 < x_2 \Rightarrow E(x_1) < E(x_2)$

Also it follows from (7) that

$E(-x_2) < E(-x_1)$ when $-x_2 < -x_1 <$

Hence the fn E is strictly increasing from 0 to $+\infty$ on the whole real line.

Define cosine & sine fns as sums of P.S.

P.T. (i). $S(x+y) = S(x)C(y) + C(x)S(y)$.

(ii). $C(x+y) = C(x)C(y) - S(x)S(y)$

where C, S denote cosine & sine respectively.

Pf: The cosine fn is defined as

$$C(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \quad \forall x \in \mathbb{R} \quad (1)$$

Sine fn is defined as

$$S(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots \quad \forall x \in \mathbb{R} \quad (2)$$

These two P.S. are uniformly convergent $\forall x \in \mathbb{R}$.

and consequently

The fns $C(x)$ and $S(x)$ are continuous $\forall x \in \mathbb{R}$.

Differentiating term by term the convergent series (2)

$$S'(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

This gives the uniformly convergent series (1)

$\therefore\ S(x)$ is derivable $\forall\ x \in \mathbb{R}$

and $\boxed{S'(x) = C(x)}$ ——— ③

Similarly $\boxed{C'(x) = -S(x)}$ ——— ④

from ① & ②

$$\left.\begin{aligned} S(0) &= 0 \\ C(0) &= 1 \end{aligned}\right] \text{ ——— ⑤}$$

Let y be any arbitrary but fixed real no

We write

$$f(x) = S(x+y) - S(x)C(y) - C(x)S(y) \quad \text{——— ⑥}$$

$$g(x) = C(x+y) - C(x)C(y) + S(x)S(y) \quad \text{——— ⑦}$$

Differentiating w.r.to x.

$$-\cancel{[C(x)C(y) + S(x)S(y)]} + [-S(x)S(y) + \cancel{C(x)C(y)}]$$

$$f'(x) = C(x+y) - C(x)C(y) + S(x)S(y) = g(x)$$

$$g'(x) = -S(x+y) + S(x)C(y) + C(x)S(y) = -f(x)$$

$$\frac{d}{dx}\left[f^2(x)+g^2(x)\right] = 2f(x)f'(x) + 2g(x)g'(x)$$
$$= 2f(x)g(x) - 2g(x)f(x)$$
$$= 0 \quad \forall x$$

$\Rightarrow f^2(x)+g^2(x)$ is a constant $\forall x$

$$\Rightarrow f^2(x)+g^2(x) = f^2(0)+g^2(0) \quad \forall x$$

$\begin{pmatrix} S(0)=0 \\ C(0)=1 \end{pmatrix}$ $= [S(y) - S(0)C(y) - C(0)S(y)]^2$

$+ [C(y) - C(0)C(y) + S(0)S(y)]^2$

$= 0$ [using (5)]

$\therefore f^2(x)+g^2(x) = 0 \ \forall x \in \mathbb{R}$

$\Rightarrow f(x) = 0$ and $g(x) = 0$

(6) & (7) $\Rightarrow$

$$\left.\begin{aligned} S(x+y) &= S(x)C(y) + C(x)S(y) \\ C(x+y) &= C(x)C(y) - S(x)S(y) \end{aligned}\right] \circledast$$

Cor 1 : $S(-x) = -S(x)$
$c(-x) = c(x) \quad \forall\, x \in \mathbb{R}$

$\because$ we have $S(-x) = -x - \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} \cdots$

$$= -\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$= -S(x).$$

Similarly $\boxed{c(-x) = c(x)}$

Cor 2 : $S(x-y) = S(x)c(y) - c(x)S(y)$
$c(x-y) = c(x)c(y) + S(x)S(y)$

Pf: Replace y by $-y$ in (the First Thm)
$\Rightarrow$ and using Cor - 1.

Cor 3 : $c^2(x) + S^2(x) = 1 \quad \forall\, x \in \mathbb{R}$

we have $1 = c(0) = c(x-x)$

$$= c(x)c(x) - S(x)S(x)$$

$$= c^2(x) + S^2(x)$$

$\therefore$ $c^2(x) + S^2(x) = 1$

Cor 4 $|c(x)| \le 1, \quad |S(x)| \le 1$

$c^2(x) + S^2(x) = 1 \quad \forall\, x$

$\Rightarrow |S(x)| \le 1, \quad |c(x)| \le 1 \quad \forall\, x$

$\sin 2x = 2\sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$.

Replacing y by x in (i) $\left[\begin{array}{l} S(x+y) = S(x)C(y) \\ \quad + C(x)S(y) \\ C(x+y) = C(x)C(y) \\ \quad - S(x)S(y) \end{array}\right.$

$S(2x) = 2S(x)C(x)$

$\boxed{C(2x) = C^2(x) - S^2(x)}$

It may be noted that the above prop. of $C(x)$ and $S(x)$ are similar to those of the trigonometric fns $\cos x$, $\sin x$ resp.

Thus we have $\boxed{C(x) \equiv \cos x}$

$\boxed{S(x) \equiv \sin x}$