# Relations. ch-3 (Liu).

## Cartesian Product $(A \times B)$

$$\{a,b\} \times \{a,c,d\} = \{(a,a),(a,c),(a,d),(b,a),(b,c),(b,d)\}$$

**Binary relation** from $A \times B$ is a subset of $A \times B$.

eg:- $A = \{a,b,c,d\}$, $B = \{\alpha, \beta, \gamma\}$ &

let $R = \{(a,\alpha),(b,\gamma),(c,\alpha),(c,\gamma),(d,\beta)\}$ be a binary Relation from A to B.

Rep. in tabular form

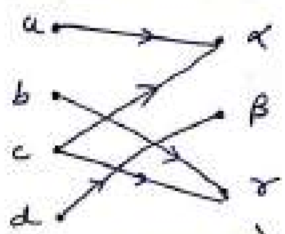
| | $\alpha$ | $\beta$ | $\gamma$ |
|---|---|---|---|
| a | ✓ | | |
| b | | | ✓ |
| c | ✓ | | ✓ |
| d | | ✓ | |

matrix form

$$M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(matrix for Rel. R)

graphical form




(digraphs).

# Let $R_1$ & $R_2$ be two binary Relations from A to B. Then $R_1 \cap R_2$ (Intersection), $R_1 \cup R_2$ (union), $R_1 \oplus R_2$ (Ringsum), $R_1 - R_2$ are also binary relations from A to B.

# ternary Relation among three sets A, B & C is defined as a subset of the cartesian product of the two sets $A \times B$ and C i.e. $(A \times B) \times C$

for eg:- $A = \{a,b\}$, $B = \{\alpha, \beta\}$, $C = \{1,2\}$.

$(A \times B) = \{(a,\alpha),(a,\beta),(b,\alpha),(b,\beta)\}$

$$(A \times B) \times C = \{((a,\alpha),1),((a,\alpha),2),((a,\beta),1),((a,\beta),2), ((b,\alpha),1),((b,\alpha),2),((b,\beta),1),((b,\beta),2)\}$$

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# quaternary Relation :- defined on 4 - sets.

ie $((A \times B) \times C) \times D$

# n-ary relation :- n-ary relation is defined on the sets

$A_1, A_2, \ldots, A_n$ as :-

$((A_1 \times A_2) \times A_3) \ldots \times A_n$

## Properties of Binary Relations

Binary Relation on A :- Binary Relation from a set A to A is said to be binary Relation on A.

eg:- A is +tive set of integers.

Binary Relation R on A is $(a, b)$ iff $a - b \geq 10$.

$\therefore R = \{ (12, 1)$ ✓, // $12 - 1 = 11 > 10$

$(12, 3)$ ✗, // $12 - 3 = 9 \not\geq 10$

$(1, 12)$, // $1 - 12 = -11 \not\geq 10$

⋮

$\}$

Thus elements in R are $R = \{ (11, 1), (12, 1), (13, 1), \ldots$

$(12, 2), (13, 2), (14, 2), \ldots$

$\ldots \}$.

## Reflexive Relation

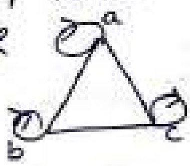
Reflexive Relation :- Let R be a binary Relation on A. R is said to be reflexive Relation if $(a,a)$ is in R for every a in A.

OR

In Reflexive Relation, every element in A is related to itself. OR

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

OR




eg. $A = \{a, b, c\}$

$R_1 = \{(a,a), (b,b), (c,c)\}$ → necessary.

$R_2 = \{(a,a), (b,b), (c,c), (a,c), (c,a), \ldots\}$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & \boxed{0} \end{bmatrix}$$

Not a Reflexive Relation

eg. $A = \{$ set of +tive integers $\}$

$R = \{(a,b)$ is in R iff a divides b $\}$

ie $R = \{(1,1)\ (2,2), (3,3), \ldots\ldots,$ // each integer divides itself.

$(2,4), (3,6), (3,9) \ldots\ldots \}$

## Symmetric Relation

Symmetric Relation :- Let R be a binary Relation on A. R is said to be symmetric Relation if $(a,b)$ in R implies that $(b,a)$ is also in R.

eg. $A = \{a, b, c\}$

$R_1 = \{(a,b), (b,a), (c,a), (a\ c)\}$

$R_2 = \{(a,a)\}$ , $R_3 = \{(a,a), (b,a), (a,b)\}$

OR

$$R_1 = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}, \quad R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

## antisymmetric Relation :-

Let R be a binary Relation on A. R is said to be antisymmetric Relation if (a, b) in R implies that (b, a) is not in R unless a = b.

OR.

If both (a, b) & (b, a) are in R, then it must be the case a = b.

eg. $A = \{a, b, c\}$

$R = \{(a, b), (a, c), (b, c), (c, c)\}$ ✓

## Transitive Relation :-

Let R be a binary relation on A. R is said to be transitive relation if ~~(a, a) is such~~ ~~whenever~~ both (a, b) & (b, c) ~~are~~ in R implies (a, c) in R

eg :- $A = \{a, b, c\}$

$R = \{(a, a), (a, b), (a, c), (b, c)\}$

checking. $(a, a), (a, b) \Rightarrow (a, b)$ ✓

$(a, a), (a, c) \Rightarrow (a, c)$ ✓

$(a, b), (b, c) \Rightarrow (a, c)$ ✓

thus R is transitive.

$A = \{a, b, c, d\}$

$R_2 = \{(a,a), (a,b), (a,c), (b,c), (d,d)\}$

$R_2$ is transitive.

$R_3 = \{(a,b)\}$, $R_3$ is transitive Relation.

$R_4 = \{(a,b), (b,c)\}^{\times}$, $R_4$ is not transitive $\because$ $(a,c)$ should belong to $R_4$.

Examples.

① $A = \{x, y, z, u, v\}$

$R = \{(x,x), (x,y), (x,v), (y,z), (z,z), (z,u), (u,u), (v,v)\}$

Reflexive = × $(\because (y,y) \notin R)$

② $A = \{2, 6, 5, 10\}$

$R = \{ aRb$ if $b$ is a multiple of $a.\}$

ie $R = \{(2,6), (2,10), (2,2), (6,6), (5,5), (5,10), (10,10)\}$

Reflexive = ✓ $(\because (2,2), (6,6), (5,5), (10,10) \in R)$

symetric = × $(\because (2,6) \in R$ but $(6,2) \notin R)$.

antisymmetric = ✓ $(\because (6,2), (10,2), (10,5) \notin R$ until $a = b)$.

③ $A = \{-2, 2, 6, 5, 10\}$

$R = \{ aRb,$ if $b$ is multiple of $a \}$

ie $R = \{(-2,2), (2,-2), (2,2), (-2,-2), \ldots\}$

antisymetric = × $\{\because$ $-2$ is multiple of $2$,
2 is " of $-2$ & $2 \neq -2$.

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eg. $S = \{p, q, r\}$

$R = \{(p,p), (q,q), (r,r), (p,q), (q,p), (q,r), (r,q)\}$

reflexive = ✓

symmetric = ✓

transitive = × ( $\because (p,q), (q,r) \in R$

but $(p,r) \notin R$ )

eg. $A = \{a, b, c\}$

$R = \{(a,b), (b,c), (a,c)\}$

Reflexive = ×

symmetric = ×

transitive = ✓

eg. $A = \{a, 3, \beta\}$

$R = \{(a,3), (3,\beta)\}$

transitive = ×

eg. $A = \{a, b, c\}$

$R = \{(b,b), (c,c), (b,c), (c,b)\}$

Reflexive = × ( $\because (a,a) \notin R$

symmetric = ✓

transitive = ✓

## 3.4. Closure of Relations

**Closure of a relation** is the smallest extension of the relation that has certain specific properties such as reflexivity, symmetry, & transitivity.

Let $R$ be a relation on set $A$.

$R$ may or may not have certain property $P$ such as reflexivity, symmetry or transitivity.

If there exist a relation $T$ with property $P$ containing $R$ such that $T$ is a subset of every relation with property $P$ containing $R$, then $T$ is known as closure of $R$ with respect to $P$.

eg. $A = \{a, b, c\}$

$R = \{(a,b)\ (b,a)\ (b,b)\ (c,b)\}$

# $R$ is not reflexive. Thus $P$ = reflexivity.

To make it reflexive, add $(a,a)$ & $(c,c)$.

$T = \{\underbrace{(a,b)\ (b,a)\ (b,b)\ (c,b)}_{R}\ (a,a)\ (c,c)\}$ // $T$ with Prop. $P$ containing $R$

$\therefore \quad R \subseteq T$

# Now, any other reflexive relation say $R''$ containing $R$ must also contain $(a,a)$ & $(c,c)$. So $T \subseteq R''$.

ie $R'' = \{\overbrace{\underbrace{(a,b), (b,a), (b,b), (c,b)}_{R}, (a,a), (c,c)}^{T}, (c,a), (a,c)\}$

$\boxed{\therefore\ T \text{ is reflexive closure of } R}$

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## Reflexive closure

A relation T is reflexive closure of a relation R iff,

(a) T is reflexive

(b) $R \subseteq T$

(c) for any relation $R''$, if $R \subseteq R''$ & $R''$ is reflexive then $T \subseteq R''$ ie. T is the smallest Relation that satisfies (a) & (b) (ie T is reflexive closure of R).

The Reflexive closure of a relation R is denoted by $r(R)$.

## Symmetric closure

A relation T is symmetric closure of a relation R iff,

(a) T is symmetric

(b) $R \subseteq T$

(c) for any relation $R''$, if $R \subseteq R''$ and $R''$ is symmetric, then $T \subseteq R''$ ie T is the smallest relation that satisfies (a) & (b). [denoted by $s(R)$].

eg. $A = \{a, b, c\}$

$R = \{(a,a)\ (a,b)\ (c,c)\ (b,c)\ (b,a)\ (a,c)\}$. R is not symmetric

To make it symmetric add $(c,b)$ & $(c,a)$

$\therefore T = \{\underbrace{(a,a)\ (a,b)\ (c,c)\ (b,c)\ (b,a)\ (a,c)}_{R},\ (c,b).(c,a)\}$

$\therefore R \subseteq T$. Here T is symmetric.

Now any other symmetric relation say $R''$ contains R must also contain (c, b) & (c, a).

Now T contains R ($R \subseteq T$), it is symmetric & it is contained within every symmetric relation that contains R, ie $T \subseteq R''$. So T is symmetric closure of R.

## Transitive closure

A relation T is the transitive closure of a relation R iff

(a) T is transitive

(b) $R \subseteq T$

(c) for any relation $R''$, if $R \subseteq R''$ & $R''$ is transitive, then $T \subseteq R''$ ie T is the smallest relation that satisfies (a) and (b).

denoted by $t(R)$.

eg. $A = \{1, 2, 3, 4, 5, 6\}$

$R = \{(1,2)\ (1,4)\ (2,4)\ (4,3)\ (5,6)\}$. # R is not transitive. To make it transitive, we have to add (1,3) (2,3).

ie $T = \{\underbrace{(1,2)\ (1,4)\ (2,4)\ (4,3)\ (5,6)}_{R}, (1,3), (2,3)\}$

$R \subseteq T$, T is transitive (a & b fulfilled).

$$R'' = \{\overbrace{\underbrace{(1,2)\ (1,4)\ (2,4)\ (4,3)\ (5,6)}_{R}\ (1,3)\ (2,3)}^{T}, (2,2)\}$$

Here $R \subseteq R''$ & $R''$ is transitive, then $T \subseteq R''$.

$\therefore$ T is the transitive closure of R.

# Partial ordering Relations (3.7).

A binary Relation is said to be a partial ordering relation if it is reflexive, antisymmetric & transitive.

For eg:-

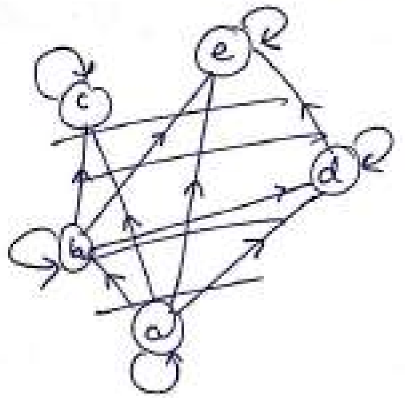
$$\begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

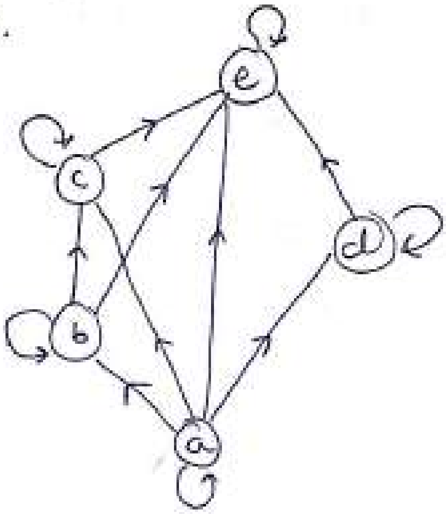
$A = \{a, b, c, d, e\}$

## # Alternate graphical Representation

Represent elements in A by Points & use arrows to represent the ordered pairs in R.

ie



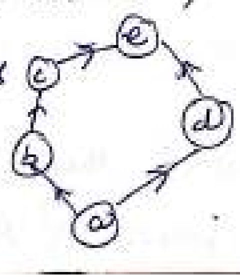

further extension :-

→ Since the relation is understood to be reflexive, omit the selfloops.

→ since the relation is understood to be transitive, omit arrows b/w points that are connected by sequences of arrows.

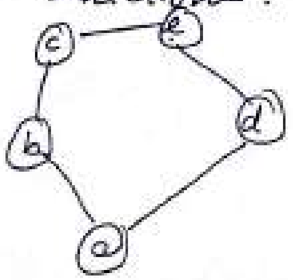
ie (a,b) & (b,c) exists, omit (a,c). & so on.

the graph becomes



If all the arrows point in one direction (ie upward, downward, left to right, right to left), we can also omit the directions.

ie



Such a graphical representation of a partial ordering relation in which all the arrowheads are understood to be pointing upward is known as <u>Hasse diagram</u> of the relation.

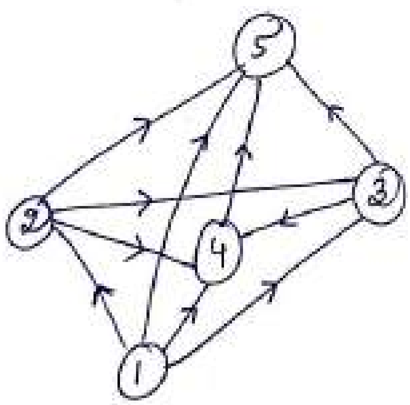
<u>eg.</u> Draw Hasse diagram for the Relation $R$ on $A = \{1, 2, 3, 4, 5\}$ whose relation matrix is,

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

<u>Sol$^n$.</u>

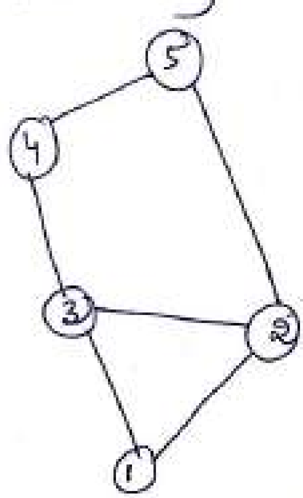
Relation R is thus,

$R = \{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (5,5),\ (1,2)\ (1,3)\ (1,4)\ (1,5)\ (2,3)\ (2,4)\ (2,5)\ (3,4)\ (3,5)\ (4,5)\}$



(omit the loops)

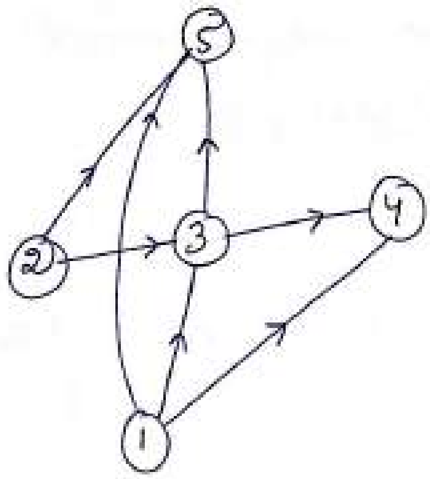
omit the transitive Rel$^n$ $\longrightarrow$



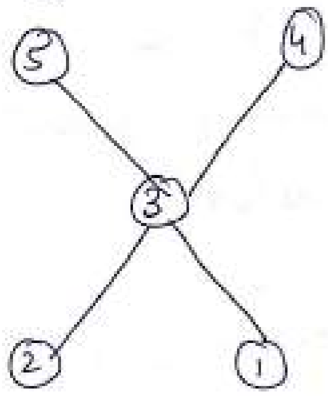
eg. Draw Hasse diagram for the Relation R on $A = \{1, 2, 3, 4, 5\}$, whose Relation matrix is

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

sol$^n$ Here $R = \{(1,1)\ (1,3)\ (1,4)\ (1,5)\ (2,2)\ (2,3)\ (2,4)\ (2,5),\ (3,3),\ (3,4)\ (3,5),\ (4,4),\ (5,5)\}$




digraph.




Hasse diagram

## Partially ordered set

Set A, together with a partial ordering Relation R on A, is called partially ordered set. [in above eg:- A = {1, 2, 3, 4, 5} is called partially ordered set].

→ Denoted by (A, R)

→ Abbreviated as poset.

→ for each ordered pair (a, b) in R, we write $a \preceq b$ instead of $(a, b) \in R$, where $\preceq$ is a generic symbol corresponding to the set of ordered pairs R [Read as less than or equal to]

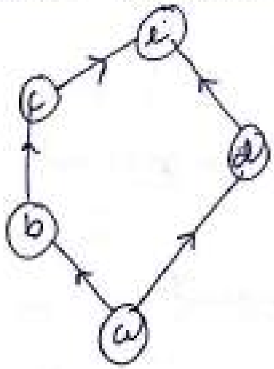
→ Partially ordered set is usually denoted by $(A, \preceq)$

# Let $(A, \preceq)$ be a partially ordered set.

Chain :- A subset of A is called a 'chain' if every two elements in the subset are related.

antichain :- A subset of A is called 'antichain' if no two distinct elements in the subset are related.

Eg:-

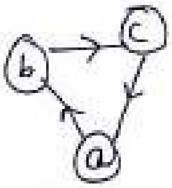



chains: {a, b, c, e} {a, b, c}, {a, b} {a}, {b}, {d, e}
{a, d, e} {a, d}, {b, c, e} {b, c}
{c, e}

antichains: {b, d} {c, d}, {d, b}, . . . .

## Totally ordered set

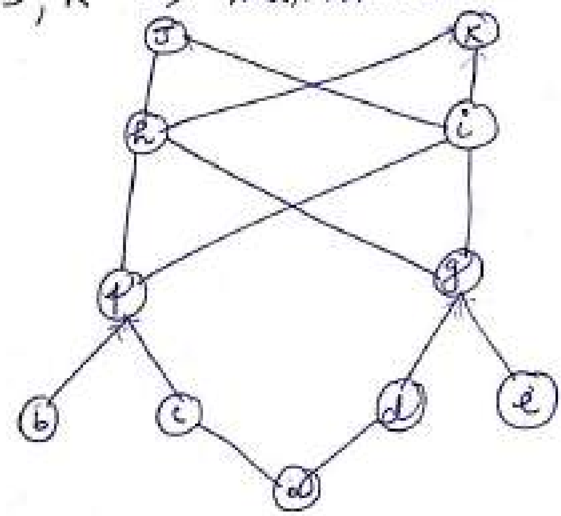
A partially ordered set $(A, \leq)$ is called totally ordered set if $A$ is a chain.



# Let $(A, \leq)$ be a partially ordered set.

**Maximal element** :- An element $a$ in $A$ is called maximal element if for no $b$ in $A$, $a \neq b$, $a \leq b$.

eg: J, K → maximinal elements.



(Hassediagram)
(upward direction).

**minimal elements** :- An element $a$ in $A$ is called minimal element if for no $b$ in $A$, $a \neq b$, $b \leq a$.

eg: $a, b, e$ → minimal elements.

**Cover** :- An element a is said to cover another element b if $b \leq a$ &, for no other element c, $b \leq c \leq a$.

eg :- f covers b, f covers c but f does not cover a.

**upperbound** :- An element c is said to be upperbound of a & b if $a \leq c$ & $b \leq c$.

for eg :- h upperbound of f, g.

i " of f, g.

J, K. " " f, g.

**least upperbound** :- An element c is said to be least upperbound of a & b if c is an upperbound of a & b, & if there is no other upper-bound d of a and b such that $d \leq c$.

eg :- h is least upper bound of f, g.

i " " " " f, g.

**lower bound** :- An element c is said to be lower bound of a and b if $c \leq a$ and $c \leq b$.

eg :- a, b, c, d, e, f, g are lowerbounds of i.

**greatest lowerbounds** :- An element c is said to be greatest lowerbounds of a & b if c is a lowerbound of a and b & if there is no other lower bound d of a and b such that $c \leq d$.

eg :- f & g are greatest lowerbounds of h & i.

Q-1 (b) Prove that the given boolean Expression is a tautology using equivalence rules:

$$(\neg p \wedge q) \rightarrow (\neg((q \rightarrow p)).$$

$$(\neg p \wedge q) \rightarrow (\neg(\neg q \vee p)) \quad [\text{conditional}]$$

$$(\neg p \wedge q) \rightarrow (\neg\neg q \wedge \neg p) \quad [\text{Demorgan's law}]$$

$$\neg((\neg p \wedge q)) \vee (\neg\neg q \wedge \neg p)$$

$$\neg((\neg p \wedge q) \vee ( q \wedge \neg p)$$

$$\neg((\neg p \wedge q) \vee (\neg p \wedge q) \quad [A + \bar{A} = T]$$

$$= T$$

Q-1 (c) $f(x) = x^2+1$ , $g(x) = x+2$.

Find $f \circ g$ & $g \circ f$ where $f$ and $g$ are $f^n$ from R to R

$$f \circ g = f(g(x)) = f(x+2)$$

$$= (x+2)^2 + 1 = x^2 + 4 + 4x + 1$$

$$= x^2 + 4x + 5$$

$$g \circ f = g(f(x) = g(x^2+1)$$

$$= (x^2+1) + 2 = x^2 + 3$$