$\int_0$

Shift operator :-

Let $y_0, y_1, \ldots y_n$ be the value of function at $x_0, x_1, \ldots x_n$ where x_i's are equally spaced points.

Then, E is called shifting operator and defined as

$$Ey_i = y_{i+1}$$

$$E^2 y_i = E(Ey_i) = E(y_{i+1}) = y_{i+2}$$

$$\Rightarrow E^2 y_i = y_{i+2}$$

$$\vdots$$

$$E^r y_i = y_{i+r}$$

$$x_{i+1} = x_i + h$$

$$y_{i+1} = f(x_{i+1}) = f(x_i + h)$$

$$y_{i-1} = f(x_{i-1}) = f(x_i - h)$$

Backward Diff. Operator :-

$$\nabla y_i = y_i - y_{i-1}$$

$$\Rightarrow \nabla y_i = f(x_i) - f(x_{i-1})$$

$$\Rightarrow \boxed{\nabla y_i = f(x_i) - f(x_i - h)}$$

Forward Diff. Operator :-

$$\Delta y_i = y_{i+1} - y_i$$

$$\Rightarrow \Delta y_i = f(x_{i+1}) - f(x_i)$$

$$\Rightarrow \boxed{\Delta y_i = f(x_i + h) - f(x_i)}$$

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Result :-

(i) $\Delta = E - 1$

$$\because \Delta f(x_i) = f(x_i + h) - f(x_i)$$

$$\Rightarrow \Delta f(x_i) = f(x_{i+1}) - f(x_i)$$

$$\Rightarrow \Delta f(x_i) = E f(x_i) - f(x_i)$$

$$\Rightarrow \boxed{\Delta = E - 1}$$

(ii) $\nabla = 1 - E^{-1}$

$$\because \nabla f(x_i) = f(x_i) - f(x_i - h)$$

$$\Rightarrow \nabla f(x_i) = f(x_i) - f(x_{i-1})$$

$$\Rightarrow \nabla f(x_i) = f(x_i) - E^{-1} f(x_i)$$

$$\Rightarrow \boxed{\nabla = 1 - E^{-1}}$$

(iii) $E\nabla = \Delta = \nabla E$

$$\because \Delta f(x_i) = f(x_{i+1}) - f(x_i)$$

$$\Rightarrow \Delta f(x_i) = f(x_i + h) - f(x_i)$$

$$\Rightarrow \Delta f(x_i) = E f(x_i) - f(x_i)$$

$$
\begin{aligned}
E\nabla(f(x_i)) &= E(\nabla(f(x_i))) \\
&= E(f(x_i) - f(x_{i-1})) \\
&= Ef(x_i) - Ef(x_{i-1}) \\
&= f(x_{i+1}) - f(x_i) = \Delta f(x_i)
\end{aligned}
$$

$$\Rightarrow E\nabla = \Delta$$

similarly $\quad \nabla E = \Delta$

$$\Rightarrow E\nabla = \Delta = \nabla E$$

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(iv) $$\Delta - \nabla = \Delta\nabla$$

$$
\begin{aligned}
\because (\Delta-\nabla)f(x_i) &= \Delta f(x_i) - \nabla f(x_i) \\
&= f(x_{i+1}) - f(x_i) - f(x_i) + f(x_{i-1}) \\
&= f(x_{i+1}) - 2f(x_i) + f(x_{i-1})
\end{aligned}
$$

$$
\begin{aligned}
(\Delta\nabla)f(x_i) &= \Delta(\nabla f(x_i)) \\
&= \Delta(f(x_i) - f(x_{i-1})) \\
&= \Delta f(x_i) - \Delta f(x_{i-1}) \\
&= f(x_{i+1}) - f(x_i) - f(x_i) + f(x_{i-1}) \\
&= f(x_{i+1}) - 2f(x_i) + f(x_{i-1})
\end{aligned}
$$

$$\Rightarrow \boxed{\Delta - \nabla = \Delta\nabla}$$

Central Difference Operator :-

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

$$\Rightarrow \delta f(x_i) = f\left(x_i+\frac{h}{2}\right) - f\left(x_i-\frac{h}{2}\right)$$

$$
\begin{aligned}
\delta^2 f(x_i) &= \delta(\delta f(x_i)) \\
&= \delta(f(x_i+h/2) - f(x_i-h/2)) \\
&= \delta f(x_i+h/2) - \delta f(x_i-h/2) \\
&= f(x_i+h) - f(x_i) - f(x_i) + f(x_i-h)
\end{aligned}
$$

$$\Rightarrow \boxed{\delta^2 f(x_i) = f(x_i+h) - 2f(x_i) + f(x_i-h)}$$

Average Difference Operator :-

$$\mu f(x) = \frac{f(x+h/2) + f(x-h/2)}{2}$$

i.e. $$\mu f(x_i) = \frac{f(x_i+h/2) + f(x_i-h/2)}{2}$$

Now, $$\mu^2 f(x_i) = \mu(\mu f(x_i))$$
$$= \frac{\mu}{2}\left[f(x_i+h/2) + f(x_i-h/2)\right]$$

$$\Rightarrow \mu^2 f(x_i) = \frac{1}{2}\left[\mu f(x_i+h/2) + \mu f(x_i-h/2)\right]$$

$$= \frac{1}{2}\left[\frac{f(x_i+h)+f(x_i)}{2} + \frac{f(x_i)+f(x_i-h)}{2}\right]$$

$$\Rightarrow \boxed{\mu^2 f(x_i) = \frac{1}{4}\left[f(x_i+h) + 2f(x_i) + f(x_i-h)\right]}$$

Result :-

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(i) $$\delta = E^{1/2} - E^{-1/2}$$

(ii) $$\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$$

(iii) $$\mu = E^{1/2} - \frac{1}{2}\delta$$

(iv) $$\mu^2 = \frac{\delta^2}{4} + 1$$

(v) $$\Delta = \frac{1}{2}\delta^2 + \delta\mu$$

(vi) $$\delta = \Delta E^{-1/2} = E^{-1/2}\Delta$$

(vii) $$\delta = \nabla E^{1/2} = E^{1/2}\nabla$$

(ix) $$\delta^n f(x) = \Delta^n f\left(x - \frac{nh}{2}\right) = \nabla^n f\left(x + \frac{nh}{2}\right)$$