Numerical method :- By Kapil Kumar

Modified Euler's Method :- (1.) Heun's Method :- One method to improve the estimate of the slope involves the determination of two derivatives for the interval - one at the begining and another at the end. The two derivatives are then averaged to obtain an improved estimate of the slope for entire interval. This approach, called Heun's method. 47.

In modified Euler's method, we start with initial value of y_0, and approximate value of y_1 is calculated from Euler's method which gives the first approximation of y_1 at $x = x_1$ to this method. This first approximation is given by i.e.

$$y_1^{(1)} = y_0 + hf(x_0, y_0)$$

Consider the first interval (x_0, x_1). Now at the end of this interval, we get

$$\left(\frac{dy}{dx}\right)_1^{(1)} = F(x_1, y_1^{(1)}) \quad \left[\because \frac{dy}{dx} = F(x, y)\right]$$

Therefore, the improved value of Δy is given by $\Delta y = h$ Average value of $\frac{dy}{dx}$ in the interval (x_0, x_1)

$$= h \cdot \frac{1}{2}\left[\left(\frac{dy}{dx}\right)_{x_0} + \left(\frac{dy}{dx}\right)_{x_1}^{(1)}\right]$$

$$= \frac{h}{2}[F(x_0, y_0) + F(x_1, y_1^{(1)})]$$

Now the second approximation of y is given

$$y_1^{(2)} = y_0 + \Delta y = y_0 + \frac{1}{2}h[F(x_0, y_0) + F(x_1, y_1^{(1)})]$$

Again, the same interval (x_0, y_0), we get an approximate value of $\frac{dy}{dx}$ at x_1 by putting the improved value of $y_1^{(1)}$ of y in $\frac{dy}{dx} = f(x, y)$, which is given by i.e. $\left(\frac{dy}{dx}\right)^{(2)} = F(x_1, y_1^{(2)})$ Similarly, the third approximation of y_1 is given by

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[\left(\frac{dy}{dx}\right)_{x_0} + \left(\frac{dy}{dx}\right)_{x_1}^{(2)}\right]$$

$$= y_0 + \frac{h}{2}[F(x_0, y_0) + F(x_1, y_1^{(2)})].$$

This process continues till two consecutive approximate value of y at x_1. This taking as the starting point for the next interval (x_1, x_2) once y_1 is obtained to desired degree of accuracy. Now repeat the above process to find better approximation of y_1, y_2 — etc. Let us take an example.

Ex:- 11.1: Given $\frac{dy}{dx} = x+y$ where $y(0)=1$. Find the value of y when $x=1$ by using Euler modified method upto 3 decimal places taking $h=0.5$.

Soln: First by using Euler method $y_1 = y_0 + h\, f(x_0, y_0)$

$y_1 = 1 + 0.5(x_0+y_0) \Rightarrow y_1 = 1+0.5(0+1) = 1.5$

By using Euler's modified method i.e

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1,y_1)] \quad x_1 = x_0+h = 0+0.5 = 0.5$$

$$y_1^{(1)} = 1+\frac{0.5}{2}[(0+1) + (0.5+1.5)] = 1.75$$

$$y_1^{(2)} = 1+\frac{0.5}{2}\{(0+1) + (0.5+1.75)\} = 1.8125$$

$$y_1^{(3)} = 1+\frac{0.5}{2}[(0+1) + (0.5+1.8125)] = 1.828125$$

$$y_1^{(4)} = 1+\frac{0.5}{2}[(0+1)+(0.5+1.828125)] = 1.83203125$$

$$y_1^{(5)} = 1+\frac{0.5}{2}[(0+1) + (0.5+1.83203125)] = 1.833007813$$

$$y_1^{(6)} = 1+\frac{0.5}{2}[(0+1) + (0.5) + 1.833007813)] = 1.833251953$$

Now as we can see that there is a minor difference in 5^{th} & 6^{th} iteration therefore $y_1 = 1.833$

Now for $y_2 = y_1 + h[f(x_1,y_1)]$, $y_1 = 1.833$, $x_1 = 0.5$

$$y_2 = 1.833 + 0.5(0.5+1.833) = 2.9995$$

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1,y_1) + f(x_2,y_2)], \quad x_2 = x_1+h = 0.5+0.5 = 1$$

$$y_2^{(1)} = 1.833 + \frac{0.5}{2}[(0.5+1.833) + (1+2.9995)] = 3.415$$

$$y_2^{(2)} = 1.833 + \frac{0.5}{2}[(0.5+1.833)+(1+3.415)] = 3.520$$

$y_2^{(3)} = 1.833 + \frac{0.5}{2}[(0.5+1.833) + (1+3.520)] = 3.54625$

$y_2^{(4)} = 1.833 + \frac{0.5}{2}[(0.5+1.833) + (1+3.54625)] = 3.5528125$

$y_2^{(5)} = 1.833 + \frac{0.5}{2}[(0.5+1.833) + (1+3.5528125)] = 3.5544531\,2$

$y_2^{(6)} = 1.833 + \frac{0.5}{2}[(0.5+1.833) + (1+3.55445312)] = 3.55486328$

As the difference between 5^{th} iteration & 6^{th} iteration is very minor

Therefore $y_2 = 3.554$ so $y(1) = 3.554$.

Ex2-(2): Apply Euler's modified method to approximate the solution of the initial value problem and calculate $y(0.3)$ by using $h = 0.3$

$$\frac{dy}{dx} = 1 + xy,\ y(0) = 2.$$

Soln:- Here $\frac{dy}{dx} = f(x,y) = 1 + xy$, $x_0 = 0$, $y_0 = 2$

$f(x_0,y_0) = 1 + 0\times 2 = 1$ $\quad x_1 = x_0 + h = 0 + 0.3 = 0.3$

Now $y_1^{(0)} = y_0 + h f(x_0,y_0) = 2 + 0.3(1 + 0\times 2) = 2 + 0.3 = 2.3$

Now using modified Euler's method is

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1,y_1)]$ $\quad x_1 = x_0 + h = 0 + 0.3 = 0.3$

$y_1^{(1)} = 2 + \frac{0.3}{2}[1 + 1 + 0.3\times 2.3] = 2 + \frac{0.3}{2}[2 + 0.69] = 2.4$

$y_1^{(2)} = 2 + \frac{0.3}{2}[1 + 1 + 0.3\times 2.4] = 2 + \frac{0.3}{2}[2 + 0.72] = 2.408$

$y_1^{(3)} = 2 + \frac{0.3}{2}[1 + 1 + 0.3\times 2.408] = 2 + \frac{0.3}{2}[2\ \cancel{+0.722}\,4]$

$= 2 + \frac{0.3}{2}[2.7224] = 2.408$

So $y_1^{(2)} = y_1^{(3)} = 2.408$

Therefore $y(0.3) = 2.408$.

Chapter-7 Ordinary Differential Equation

(Q.) **Midpoint method:-** Mid point method is a simple modification of Euler's method. The mid point method is a one-step method for numerically solving differential equations. Midpoint method uses Euler's method to predict value of y at mid-point of interval. In this method

$$y_{n+1} = y_n + F\left[x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}F(x_n, y_n)h\right]h$$

This midpoint method is superior to Euler's method because it utilize a slope estimate at the midpoint of the prediction interval.

Ex-(1.1): Given $\frac{dy}{dx} = y^2 + 5x$ when $y(0) = 2$. Taking $h = 0.5$ find the value of $y(1.5)$ using mid point method.

Soln: Here $x_0 = 0$, $y_0 = 2$, $h = 0.5$, $x_1 = x_0 + h = 0 + 0.5 = 0.5$

$F(x_0, y_0) = y_0^2 + 5x_0 = 2^2 + 5(0) = 4$, $x_0 + \frac{h}{2} = 0 + \frac{0.5}{2} = 0.25$

Using mid point formula $y_1 = y_0 + F\left[x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}F(x_0, y_0)h\right]h$

$= 2 + F[0.25,\ 2 + \frac{1}{2} \times 4 \times 0.5]\,0.5$

$= 2 + F[0.25, 3]\,0.5 = 2 + [(3)^2 + 5(0.25)]\,0.5$

$= 2 + [9 + 1.25]\,0.5 = 7.125$ Therefore $y(0.5) = 7.125$

Now $F(x_1, y_1) = y_1^2 + 5x_1 = (7.125)^2 + 5(0.5) = 53.265625$

$x_1 + \frac{1}{2}h = 0.5 + \frac{1}{2}(0.5) = 0.75$

$y_2 = y_1 + F\left[x_1 + \frac{1}{2}h,\ y_1 + \frac{1}{2}F(x_1, y_1)h\right]h$

$= 7.125 + F\left[0.75,\ 7.125 + \frac{1}{2} \times 53.265625 \times 0.5\right]0.5$

$= 7.125 + F[0.75, 20.4414062]\,0.5$

$= 7.125 + [(20.4414062)^2 + 5(0.75)]\,0.5 = 217.925544$

Therefore $y(1) = 217.925544$

Now $F(x_2, y_2) = y_2^2 + 5x_2 = (217.925544)^2 + 5(1) = 47496.5427$

$x_2 + \frac{1}{2}h = 1 + \frac{1}{2}(0.5) = 1.25$

$y_3 = y_2 + F[x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}F(x_2, y_2)h]h$

$= 217.925544 + F[1.25, 217.925544 + \frac{1}{2} \times 47496.5427 \times 0.5] 0.5$

$= 217.925544 + F[1.25, 12092.0612] 0.5 = 73109193.1$

Therefore $y(1.5) = 73109193.1$

Exi-(2.1): Given $\frac{dy}{dx} = x + y^2$, $y(0) = 1$ and $h = 0.1$ Find $y(0.1)$ using mid point method.

Soln: Here $x_0 = 0$, $y_0 = 1$ $h = 0.1$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

$F(x_0, y_0) = x_0 + y_0^2 = 0 + (1)^2 = 1$

$x_0 + \frac{h}{2} = 0 + \frac{0.1}{2} = 0.05$

Using mid point formula $y_1 = y_0 + F[x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}F(x_0, y_0)h] \times$

$= 1 + F[0.05, 1 + \frac{1}{2} \times 1 \times 0.1] 0.1$

$= 1 + F[0.05, 1 + 0.05] 0.1$

$= 1 + F[0.05, 1.05] 0.1$

$= 1 + [0.05 + (1.05)^2] 0.1$

$= 1.11525$

Therefore $y(0.1) = 1.11525$.

Runge-Kutta methods :- Runge-Kutta (RK) methods achieve the accuracy of a Taylor series approach without requiring the calculation of higher derivatives. Many variations exist but all can be cast in the generalized form

$$y_{i+1} = y_i + \phi h \quad -(1)$$

where ϕ is called an increment function, which can be interpreted as a representative slope over the interval. The increment function can be written in general form as

$$\phi = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n \quad -(2)$$

where the a's are constant & k's are

$$\left.\begin{aligned}
k_1 &= f(t_i, y_i) \\
k_2 &= f(t_i + p_1 h, y_i + q_{11} k_1 h) \\
k_3 &= f(t_i + p_2 h, y_i + q_{21} k_1 h + q_{22} k_2 h) \\
&\vdots \\
k_n &= f(t_i + p_{n-1} h, y_i + q_{n-1,1} h + q_{n-1,2} k_2 h + \cdots q_{n-1,n-1} k_{n-1} h)
\end{aligned}\right\} \quad -(3)$$

where the p's and q's are constants. k's are recurrence relationship i.e k_1 appears in the equation for k_2, which appears in the equation for k_3 and so forth. Because each k is a functional evaluation, this recurrence makes RK methods efficient for computer calculations.

Second-Order Runge-Kutta Methods :- The second order version of eq^n (1) is

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2) h \quad -(4)$$

where

$$\left.\begin{aligned}
k_1 &= f(t_i, y_i) \\
k_2 &= f(t_i + p_1 h, y_i + q_{11} k_1 h)
\end{aligned}\right\} \quad -(5)$$

The values for a_1, a_2, p_1 and q_{11} are evaluated by setting eq^n (4) equal to a second-order Taylor series. By doing this, three equation can be derived to evaluate the four unknown constants. The three equations are

$$\left.\begin{aligned} a_1 + a_2 &= 1 \\ a_2 P_1 &= 1/2 \\ a_2 q_{11} &= 1/2 \end{aligned}\right\} \quad -(6)$$

Because we have three equations with four unknowns, these equations are said to be underdetermined. We therefore must assume a value of one of the unknowns to determine the other three. Suppose that we specify a value for a_2 - Then

$$\left.\begin{aligned} a_1 &= 1 - a_2 \\ P_1 &= q_{11} = \frac{1}{2a_2} \end{aligned}\right\} \quad -(7)$$

Because we can choose an infinite number of values for a_2, there are an infinite number of second-order RK methods.

(i.) Heun method without Iteration:- If we take $a_2 = 1/2$ (let) then eq^n (7) gives

$$a_1 = 1/2 \text{ and } P_1 = q_{11} = 1$$

Now eq^n (4) & (5) becomes

$$\left.\begin{aligned} y_{i+1} &= y_i + \left(\tfrac{1}{2}k_1 + \tfrac{1}{2}k_2\right)h \\ \text{where } k_1 &= f(t_i, y_i) \\ k_2 &= f(t_i + h, y_i + k_1 h) \end{aligned}\right\} \quad -(8)$$

Here k_1 is the slope at the begining of the interval & k_2 is the slope at the end of the interval. Consequently, this second-order Runge-Kutta method is actually Heun's technique without iteration of the corrector.

(2) <u>The mid point method</u> :- ($a_2 = 1$) If a_2 is assumed to be 1 then from eqn (7) we get

$a_1 = 0$ and $P_1 = Q_{11} = \frac{1}{2}$ Now eqn (4) & (5) gives

$$\left.\begin{aligned} y_{i+1} &= y_i + k_2 h \\ \text{where} \quad k_1 &= f(t_i, y_i) \\ k_2 &= f(t_i + h/2, y_i + k_1 h/2) \end{aligned}\right\} \quad -(9)$$

This is mid point method.

(3) <u>Ralston's method</u> :- ($a_2 = 2/3$) If a_2 is assumed to be $2/3$ then from eqn (7) we get

$a_1 = \frac{1}{3}$ and $P_1 = Q_{11} = 3/4$ Now eqn (4) & (5) becomes

$$\left.\begin{aligned} y_{i+1} &= y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h \\ \text{where} \quad k_1 &= f(x_i, y_i) \\ k_2 &= f\left(t_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right) \end{aligned}\right\} \quad -(10)$$

This is Ralston's method.

<u>Ex:-(11)</u>: Given the equation $\frac{dy}{dx} = \frac{2y}{x}$ with $y(1) = 2$. Estimate $y(2)$ using Heun's method, mid-point method and Ralston method with $h = 0.25$.

<u>Soln</u>:- Using Heun's method :-

Iteration 1 Here $x_0 = 1, y_0 = 2, h = 0.25$

$k_1 = f(x_0, y_0) = \frac{2y_0}{x_0} = \frac{4}{1} = 4$

$k_2 = f(x_0 + h, y_0 + k_1 h) = f(1.25, 3)$

$= \frac{2(3)}{1.25} = 4.8$

Now $y_1 = y_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$

$y_1 = 2 + \frac{1}{2}(4+4.8)\times 0.25 = 3.1$

Iteration 2:- $x_1 = x_0 + h = 1+0.25 = 1.25$, $y_1 = 3.1$

$K_1 = f(x_1, y_1) = \frac{2y_1}{x_1} = \frac{2(3.1)}{1.25} = 4.96$

$K_2 = f(x_1 + h, y_1 + K_1 h) = f(1.5, 4.34) = \frac{2(4.34)}{1.5} = 5.78$

$y_2 = y_1 + \frac{1}{2}(K_1 + K_2)h = 3.1 + \frac{1}{2}(4.96 + 5.78)\times 0.25 = 4.44$

Iteration 3:- $x_2 = x_1 + h = 1.25 + 0.25 = 1.50$, $y_2 = 4.44$

$K_1 = f(x_2, y_2) = \frac{2y_2}{x_2} = \frac{2\times 4.44}{1.50} = 5.92$

$K_2 = f(x_2 + h, y_2 + K_1 h) = f(1.75, 5.92) = \frac{2(5.92)}{1.75} = 6.76$

$y_3 = y_2 + \frac{1}{2}(K_1 + K_2)h = 4.44 + \frac{1}{2}(5.92 + 6.76)\times 0.25 = 6.03$

Similarly we can find $y(2) = 7.86$

Using mid-Point method:- Iteration 1:- $x_0 = 1, y_0 = 2$ $h = 0.25$

$K_1 = f(x_0, y_0) = \frac{2y_0}{x_0} = \frac{4}{1} = 4$

$K_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1 h}{2}\right) = f\left(1 + \frac{0.25}{2}, 2 + \frac{4\times 0.25}{2}\right) = f(1.125, 2.5)$

$= \frac{2(2.5)}{1.125} = 4.44$

Now $y_1 = y_0 + K_2 h = 2 + 4.44\times 0.25 = 3.11$

Iteration 2:- $x_1 = x_0 + h = 1 + 0.25 = 1.25$, $y_1 = 3.11$

$K_1 = f(x_1, y_1) = \frac{2y_1}{x_1} = \frac{2(3.11)}{1.25} = 4.976$

$K_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1 h}{2}\right) = f\left(1.25 + \frac{0.25}{2}, 3.11 + \frac{4.976\times 0.25}{2}\right)$

$= f(1.375, 3.732) = \frac{2(3.732)}{1.375} = 5.428$

$y_2 = y_1 + K_2 h = 3.11 + 5.428 \times 0.25 = 4.467$

Iteration 3:- $x_2 = x_1 + h = 1.25 + 0.25 = 1.50$, $y_2 = 4.467$, $h = 0.25$

$K_1 = f(x_2, y_2) = \frac{2y_2}{x_2} = \frac{2 \times 4.467}{1.50} = 5.956$

$K_2 = f\left(x_2 + \frac{h}{2}, y_2 + \frac{K_1 h}{2}\right) = f\left(1.50 + \frac{0.25}{2}, 4.467 + \frac{5.956 \times 0.25}{2}\right)$

$= f(1.625, 5.2115) = \frac{2(5.2115)}{1.625} = 6.41415$

$y_3 = y_2 + K_2 h = 4.467 + 6.41415 \times 0.25 = 6.07$

Similarly we can find $y(2) = 7.3996$.

Using Ralston's method:- Iteration 1:- $x_0 = 1$, $y_0 = 2$, $h = 0.25$

$K_1 = f(x_0, y_0) = \frac{2y_0}{x_0} = \frac{4}{1} = 4$

$K_2 = f\left(x_0 + \frac{3}{4}h, y_0 + \frac{3}{4}K_1 h\right) = f\left(1 + \frac{3}{4} \times 0.25, 2 + \frac{3}{4} \times 4 \times 0.25\right)$

$f(1.1875, 2.75) = \frac{2(2.75)}{1.1875} = 4.632$

Now $y_1 = y_0 + \frac{h}{3}(K_1 + 2K_2) = 2 + \frac{0.25}{3}(4 + 2 \times 4.632) = 3.105$

Iteration 2:- $x_1 = x_0 + h = 1 + 0.25 = 1.25$, $y_1 = 3.105$

$K_1 = f(x_1, y_1) = \frac{2y_1}{x_1} = \frac{2(3.105)}{1.25} = 4.968$

$K_2 = f\left(x_1 + \frac{3}{4}h, y_1 + \frac{3}{4}K_1 h\right) = f\left(1.25 + \frac{3}{4} \times 0.25, 3.105 + \frac{3}{4} \times 4.968 \times 0.25\right)$

$f(1.4375, 4.0365) = \frac{2(4.0365)}{1.4375} = 5.616$

$y_2 = y_1 + \frac{h}{3}(K_1 + 2K_2) = 3.105 + \frac{0.25}{3}(4.968 + 2 \times 5.616) = 4.455$

Iteration 3:- $x_2 = x_1 + h = 1.25 + 0.25 = 1.50$, $y_2 = 4.455$

$K_1 = f(x_2, y_2) = \frac{2y_2}{x_2} = \frac{2 \times 4.455}{1.50} = 5.94$

$K_2 = f\left(x_2 + \frac{3}{4}h, y_2 + \frac{3}{4}K_1 h\right) = f\left(1.50 + \frac{3}{4}\times 0.25, 4.455 + \frac{3}{4}\times 5.94\times 0.25\right)$

$f(1.6875, 5.56875) = \frac{2\times 5.56875}{1.6875} = 6.6$

$y_3 = y_2 + \frac{h}{3}(K_1 + 2K_2) = 4.455 + \frac{0.25}{3}(5.94 + 2\times 6.6) = 6.05$

Similarly we can find $y(2) = 7.89$

The Tabulated result are i.e

Heun's method		Mid-point method		Ralston's method	
x	y	x	y	x	y
1	2	1	2	1	2
1.25	3.1	1.25	3.11	1.25	3.105
1.50	4.44	1.50	4.467	1.50	4.455
1.75	6.03	1.75	6.07	1.75	6.05
2.00	7.86	2.00	7.3996	2.00	7.89

Ex:-(2): Given $\frac{dy}{dx} = \frac{1+4x}{\sqrt{y}}$, $0 \le x \le 1$, $h = 0.25$, $y(0) = 1$ Solve by using Heun method & Mid-Point method.

Soln:- Here $x_0 = 0$, $y_0 = 1$, $h = 0.25$

Using Heun method:- Ist Iteration:- $x_1 = x_0 + h = 0 + 0.25 = 0.25$

$K_1 = f(x_0, y_0) = f(0,1) = \frac{1+4(0)}{\sqrt{1}} = 1$

$K_2 = f(x_0 + h, y_0 + K_1 h) = f(0.25, 1 + 1(0.25)) = f(0.25, 1.25)$

$= \frac{1+4(0.25)}{\sqrt{1.25}} = 1.7888$

$y_1 = y_0 + \frac{h}{2}(K_1 + K_2) = 1 + \frac{0.25}{2}(1 + 1.7888) = 1.3486$

2nd Iteration: $x_1 = x_0 + h = 0 + 0.25 = 0.25$, $y_1 = 1.3486$

$$k_1 = f(x_1, y_1) = f(0.25, 1.3486) = \frac{1+4(0.25)}{\sqrt{1.3486}} = 1.7222$$

$$k_2 = f(x_1+h, y_1+k_1h) = f(0.5, 1.3486 + 1.7222(0.25)) = f(0.5, 1.7792)$$

$$= \frac{1+4(0.5)}{\sqrt{1.7792}} = 2.2491$$

$$y_2 = y_1 + \frac{h}{2}(k_1+k_2) = 1.3486 + \frac{0.25}{2}(1.7222 + 2.2491) = 1.845$$

Iteration 3:- $x_2 = x_1 + h = 0.25 + 0.25 = 0.50$, $y_2 = 1.845$

$$k_1 = f(x_2, y_2) = f(0.50, 1.845) = \frac{1+4(0.50)}{\sqrt{1.845}} = 2.2086$$

$$k_2 = f(x_2+h, y_2+k_1h) = f(0.75, 1.845 + 2.2086 \times 0.25) = f(0.75, 2.39715)$$

$$= \frac{1+4(0.75)}{\sqrt{2.39715}} = 2.5835$$

$$y_3 = y_2 + \frac{h}{2}(k_1+k_2) = 1.845 + \frac{0.25}{2}(2.2086 + 2.5835) = 2.4490$$

Iteration 4:- $x_3 = x_2 + h = 0.50 + 0.25 = 0.75$, $y_3 = 2.4490$

$$k_1 = f(x_3, y_3) = f(0.75, 2.4490) = \frac{1+4(0.75)}{\sqrt{2.4490}} = 2.5560$$

$$k_2 = f(x_3+h, y_3+k_1h) = f(1, 2.4490 + 2.5560 \times 0.25) = f(1, 3.088)$$

$$= \frac{1+4(1)}{\sqrt{3.088}} = 2.8453$$

$$y_4 = y_3 + \frac{h}{2}(k_1+k_2) = 2.4490 + \frac{0.25}{2}(2.5560 + 2.8453) = 3.124$$

$y(1) = 3.124$

Using mid point method:- Iteration 1:- $x_1 = x_0 + h = 0.25$

$$k_1 = f(x_0, y_0) = f(0,1) = \frac{1+4\sqrt{0}}{\sqrt{1}} = 1$$

$$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + k_1\frac{h}{2}\right) = f\left(0.125, 1 + 1 \times \frac{0.25}{2}\right) = f(0.125, 1.125)$$

$$= \frac{1+4(0.125)}{\sqrt{1.125}} = 1.4142$$

$$y_1 = y_0 + k_2 h = 1 + 1.4142(0.25) = 1.35355$$

Iteration (2):- $x_1 = x_0 + h = 0.25$, $y_1 = 1.35355$

$$k_1 = f(x_1, y_1) = f(0.25, 1.35355) = \frac{1+4(0.25)}{\sqrt{1.35355}} = 1.7191$$

$$k_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1 h}{2}\right) = f\left(0.25 + \frac{0.25}{2}, 1.35355 + \frac{1.7191 \times 0.25}{2}\right)$$

$$f(0.375, 1.5684) = \frac{1+4(0.375)}{\sqrt{1.5684}} = 1.9963$$

$$y_2 = y_1 + k_2 h = 1.35355 + 1.9963 \times 0.25 = 1.8526$$

Iteration (3):- $x_2 = x_1 + h = 0.50$, $y_2 = 1.8526$

$$k_1 = f(x_2, y_2) = f(0.50, 1.8526) = \frac{1+4(0.50)}{\sqrt{1.8526}} = 2.2041$$

$$k_2 = f\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1 h}{2}\right) = f\left(0.50 + \frac{0.25}{2}, 1.8526 + \frac{2.2041 \times 0.25}{2}\right)$$

$$f(0.625, 2.1281) = \frac{1+4(0.625)}{\sqrt{2.1281}} = 2.3992$$

$$y_3 = y_2 + k_2 h = 1.8526 + 2.3992 \times 0.25 = 2.4524$$

Iteration (4):- $x_3 = x_2 + h = 0.50 + 0.25 = 0.75$, $y_3 = 2.4524$

$$k_1 = f(x_3, y_3) = f(0.75, 2.4524) = \frac{1+4(0.75)}{\sqrt{2.4524}} = 2.5542$$

$$k_2 = f\left(x_3 + \frac{h}{2}, y_3 + \frac{k_1 h}{2}\right) = f\left(0.75 + \frac{0.25}{2}, 2.4524 + 2.5542 \times \frac{0.25}{2}\right)$$

$$f(0.875, 2.75895) = \frac{1+4(0.875)}{\sqrt{2.75895}} = 2.709$$

$$y_4 = y_3 + k_2 h = 2.4524 + 2.709 \times 0.25 = 3.1296$$

Therefore $y(1) = 3.1296$