The characteristic polynomial equation is

$$P_A(x) = |A - xI| = \begin{vmatrix} 5-x & -2 & 6 & -1 \\ 0 & 3-x & -8 & 0 \\ 0 & 0 & 5-x & 4 \\ 0 & 0 & 0 & 1-x \end{vmatrix}$$

$$= (5-x)\left[(3-x)(5-x)(1-x) + 0 + 0\right] - 0 + 0 - 0$$

$$= (5-x)(3-x)(5-x)(1-x)$$

$$= (5-x)^2(3-x)(1-x)$$

$$= (25 + x^2 - 10x)(3 - 4x + x^2)$$

$$= 75 - 100x + 25x^2 + 3x^2 - 4x^3 + x^4 - 30x + 40x^2 - 10x^3$$

$$= x^4 - 14x^3 + 68x^2 - 130x + 75$$

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$$
\begin{aligned}
T(\alpha v_1) &= T[\alpha(x_1, x_2, x_3)] \\
&= T[\alpha x_1, \alpha x_2, \alpha x_3] \\
&= a(\alpha x_1) + b(\alpha x_2) + c(\alpha x_3) \\
&= \alpha[a x_1 + b x_2 + c x_3] \\
&= \alpha T((x_1, x_2, x_3)) \\
&= \alpha T(v_1)
\end{aligned}
$$

$\Rightarrow$ T is a linear transformation.

iv) $T: M_{22} \to M_{22}$ defined by

$$T\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

<u>Sol</u> We can show that the transformation is linear :-
Let $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

i)
$$
\begin{aligned}
T(A_1 + A_2) &= T\left[\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right] \\
&= T\left[\begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}\right] \\
&= \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}\left\{\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right\} \\
&= \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \\
&= T(A_1) + T(A_2)
\end{aligned}
$$

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ii) $T[\alpha A_1] = T\left[\alpha\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\right]$

$= T\left[\begin{pmatrix} \alpha a_1 & \alpha b_1 \\ \alpha c_1 & \alpha d_1 \end{pmatrix}\right]$

$= \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} \alpha a_1 & \alpha b_1 \\ \alpha c_1 & \alpha d_1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \alpha \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$

$= \alpha\left\{\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right\}$

$= \alpha T(A_1)$

$\Rightarrow$ T is a linear transformation.

v) $T: P_1 \to P_2$ defined by

$$T(ax+b) = \frac{ax^2}{2} + bx$$

Sol We can show that the transformation is linear :-

Let $p_1 = ax_1 + b \quad p_2 = ax_2 + b.$

i) $T(p_1 + p_2) = T[(ax_1+b) + (ax_2+b)]$

$= T[ax_1 + ax_2 + b + b]$

$= T[a(x_1+x_2) + b($

6. Distributivity :-

Let $a, b \in R$ and $v_1 = (x_1, x_2) \in V$ such that

$$
\begin{aligned}
(a+b)\,v_1 &= (a+b)[(x_1, x_2)] \\
&= [(a+b)x_1, (a+b)x_2] \\
&= [ax_1 + bx_1, ax_2 + bx_2] \\
&= [(ax_1, ax_2) + (bx_1, bx_2)] \\
&= [a(x_1, x_2) + b(x_1, x_2)] \\
&= a v_1 + b v_1
\end{aligned}
$$

7. Associativity of scalar Multiplication :-

Let $a, b \in R$ and $v_1 \in V$. then $v_1 = [x_1, x_2]$

$$
\begin{aligned}
(ab)\,v_1 &= (ab)[x_1, x_2] \\
&= [abx_1, abx_2] \\
&= [a(bx_1, bx_2)] \\
&= a[bx_1, bx_2] \\
&= a(bv_1)
\end{aligned}
$$

Identity under scalar Multiplication :-

Suppose $v_1 \in V$ by the definition of scalar multiplication in V and the identity axiom of R, we have

$$
\begin{aligned}
1v &= v \\
1(x_1, x_2) &= (x_1, x_2)
\end{aligned}
$$

Thus, all axioms are satisfied, and we can say that V is a vector-space.

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Q Let V = {n×n matrices with positive entries} with the usual matrix operations. (entries must be real to be compared to zero)

Sol We need to prove that V is a vector space, given the usual matrix operations. Now, we need to proof the other vector space axioms :-

1 Commutative law of addition :-
Let $V_1 = [a_{ij}]_{n\times n}$ and $W_1 = [b_{ij}]_{n\times n}$ matrix with positive entries

$$V_1 + W_1 = [a_{ij}]_{n\times n} + [b_{ij}]_{n\times n}$$
$$= [b_{ij}]_{n\times n} + [a_{ij}]_{n\times n}$$
$$= W_1 + V_1$$

Thus, the commutative law of addition holds.

2 Associative law of addition :-
Let $V_1 = [a_{ij}]_{n\times n}$ $W_1 = [b_{ij}]_{n\times n}$ $Z_1 = [c_{ij}]_{n\times n}$ matrix with positive entries.

$$(V_1 + W_1) + Z_1 = ([a_{ij}]_{n\times n} + [b_{ij}]_{n\times n}) + [c_{ij}]_{n\times n}$$
$$= [a_{ij}]_{n\times n} + [b_{ij}]_{n\times n} + [c_{ij}]_{n\times n}$$
$$= [a_{ij}]_{n\times n} + ([b_{ij}]_{n\times n} + [c_{ij}]_{n\times n})$$
$$= V_1 + (W_1 + Z_1)$$

Thus, the associative law of addition holds

3. Existence of Additive Identity :-
Let $V_1 = [a_{ij}]_{n\times n}$, the additive identity of a matrix is (W $W_1 = [b_{ij}]_{n\times n}$
such, that

$$V_1 + W_1 = V_1$$
$$[a_{ij}]_{n\times n} + W_1 = [a_{ij}]_{n\times n}$$

$$W_1 = [a_{ij}]_{n\times n} - [a_{ij}]_{n\times n}$$
$$W_1 = 0 = [b_{ij}]_{n\times n}$$

Thus, '0' zero matrix is the additive identity

Sol We can show that the transformation is linear :-
Let $v_1 = (x_1, x_2, x_3)$ and $v_2 = (y_1, y_2, y_3)$

i) $T(v_1 + v_2) = T[(x_1, x_2, x_3) + (y_1, y_2, y_3)]$
$= T[x_1 + y_1, x_2 + y_2, x_3 + y_3]$
$= v \times [x_1 + y_1, x_2 + y_2, x_3 + y_3]$
$= v \times [(x_1, x_2, x_3) + (y_1, y_2, y_3)]$
$= v \times [x_1, x_2, x_3] + v \times [y_1, y_2, y_3]$
$= T((x_1, x_2, x_3)) + T((y_1, y_2, y_3))$
$= T(v_1) + T(v_2)$

ii) $T(\alpha v_1) = T[\alpha(x_1, x_2, x_3)]$
$= T[\alpha x_1, \alpha x_2, \alpha x_3]$
$= v \times [\alpha x_1, \alpha x_2, \alpha x_3]$
$= v \times \alpha[x_1, x_2, x_3]$
$= \alpha[v \times (x_1, x_2, x_3)]$
$= \alpha T((x_1, x_2, x_3))$
$= \alpha T(v_1)$

$\Rightarrow T: E^3 \to E^3$ is a linear transformation.

iii) $T: E^3 \to E^1$ defined by

$T((x_1, x_2, x_3)) = ax_1 + bx_2 + cx_3$ where a, b, c are fixed real numbers.

Sol We can show that the transformation is linear :-
Let $v_1 = (x_1, x_2, x_3)$ and $v_2 = (y_1, y_2, y_3)$

$T(v_1 + v_2) = T[(x_1, x_2, x_3) + (y_1, y_2, y_3)]$
$= T[x_1 + y_1, x_2 + y_2, x_3 + y_3]$
$= a(x_1 + y_1) + b(x_2 + y_2) + c(x_3 + y_3)$
$= ax_1 + ay_1 + bx_2 + by_2 + cx_3 + cy_3$
$= (ax_1 + bx_2 + cx_3) + (ay_1 + by_2 + cy_3)$
$= T((x_1, x_2, x_3)) + T((y_1, y_2, y_3))$
$= T(v_1) + T(v_2)$

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Null A is the subspace spanned by {u, v} where

$$u = \begin{bmatrix} -9 \\ 7 \\ 1 \\ 0 \end{bmatrix} \text{ and } v = \begin{bmatrix} -10 \\ 7 \\ 0 \\ 1 \end{bmatrix}$$ They both are linearly independent

Basis for Null matrix = $\left\{ \begin{bmatrix} -9 \\ 7 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ 7 \\ 0 \\ 1 \end{bmatrix} \right\}$

Q2. Find a basis for the column space of the matrix:-

$$B = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 2 & 3 \\ -1 & 0 & 1 & -2 \end{bmatrix}$$

Sol. To find the basis for the column space of the m[atrix] we will first reduce the matrix into row echelon form.

$$B = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 2 & 3 \\ -1 & 0 & 1 & -2 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the column space :- A basis is $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

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eigenspace of $\lambda_1 = 10$ $\quad [A - 10I \mid 0]$

$$\Rightarrow \left[\begin{array}{ccc|c} -5 & 4 & 2 & 0 \\ 4 & -5 & 2 & 0 \\ 2 & 2 & -8 & 0 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 + R_2 \\ R_3 \to \frac{1}{2} R_3 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} -1 & -1 & 4 & 0 \\ 4 & -5 & 2 & 0 \\ 1 & 1 & -4 & 0 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -4 & 0 \\ 4 & -5 & 2 & 0 \\ -1 & -1 & 4 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -4 & 0 \\ 0 & -9 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_2 \to -\frac{1}{9} R_2$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 \to R_1 - R_2$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \quad x_1 - 2x_3 = 0 \qquad$ let $x_3 = c \Rightarrow x_1 = x_2 = 2c$

$\quad\;\; x_2 - 2x_3 = 0 \qquad \Rightarrow \{[2c, 2c, c] : c \in \mathbb{R}\}$

$\qquad\qquad\qquad\qquad \Rightarrow \{c[2, 2, 1] : c \in \mathbb{R}\}$

For $\lambda_1 = 10$ eigenspace is $\{c[2, 2, 1] : c \in \mathbb{R}\}$

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Date:- 24·03·2020

Find the eigenvalues and eigenvectors of the matrix:-

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

The characteristic polynomial of A is $p_A(x) = |A - xI|$

$$\begin{vmatrix} 1-x & -3 & 3 \\ 3 & -(5+x) & 3 \\ 6 & -6 & 4-x \end{vmatrix}$$

$$= (1-x)[-(5+x)(4-x)+18] + 3[3(4-x)-18] + 3[-18+6(5+x)]$$
$$= (1-x)(x^2 - x + 18 - 20) + 3(12 - 3x - 18) + 3(-18 + 30 + 6x)$$
$$= (1-x)(x^2 - x - 2) + 3(-3x - 6) + 3(6x + 12)$$
$$= (1-x)(x^2 - x - 2) - 9(x+2) + 9(2x+4)$$
$$= x^2 - x - 2 - x^3 + x^2 + 2x - 9x - 18 + 18x + 36$$
$$= -x^3 + 12x + 16$$
$$= -(x-4)(x^2 + 4x + 4)$$
$$= -(x-4)(x+2)^2$$

The eigenvalues are $x = 4, -2$
at $p_A(x) = 0 \rightarrow \lambda_1 = 4, -2$

The eigenvectors corresponding to $\lambda_1 = 4$ are :-
$[A - 4I \mid 0]$

$$\Rightarrow \left[\begin{array}{ccc|c} -3 & -3 & 3 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_1 \rightarrow R_1\left(-\frac{1}{3}\right) \\ R_2 \rightarrow -\frac{1}{3} R_2 \\ R_3 \rightarrow \frac{1}{3} R_3 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -1 & 3 & -1 & 0 \\ 2 & -2 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \rightarrow R_2 + R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{array}$$

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The eigenspace for $\lambda_2 = 1$ is :-

$[A - I | 0]$

$$\Rightarrow \left[\begin{array}{ccc|c} 4 & 4 & 2 & 0 \\ 4 & 4 & 2 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right] \quad R_1 \to \frac{1}{2}R_1 \quad R_2 \to \frac{1}{2}R_2$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 2 & 1 & 0 \\ 2 & 2 & 1 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 \to \frac{1}{2}R_1$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \; x_1 + x_2 + \frac{1}{2} x_3 = 0$

let $x_2 = b$

$x_3 = c$

$\Rightarrow \; x_1 = -b - \frac{1}{2}c$

eigenspace $= \{[-b - \frac{1}{2}c, b, c] : b, c \in R\}$

$= \{b[-1, 1, 0] + c[-1/2, 0, 1] : b, c \in R\}$

$= \{b[-1, 1, 0] + c[-1, 0, 2] : b, c \in R\}$

Q3. Find the characteristic polynomial equation of :-

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & -4 & 2 & 0 \end{array}\right] \quad R_3 \to R_3 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_2 \to \frac{1}{4} R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 \to R_1 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 - \frac{1}{2} x_3 = 0$ let $x_3 = c$ then

$x_2 - \frac{1}{2} x_3 = 0$ $x_1 = x_2 = \frac{1}{2} c$

$\Rightarrow \{[\frac{c}{2}, \frac{c}{2}, c] : c \in R\} = \{c[1/2, 1/2, 1] : c \in R$

$\Rightarrow$ eigenvector $= \{[1, 1, 2]\}$

The eigenvector corresponding to $\lambda_2 = -2$

$$[A+2I \mid 0] = \left[\begin{array}{ccc|c} 3 & -3 & 3 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{array}\right] \quad \begin{array}{l} R_1 \to \frac{1}{3} R_1 \\ R_2 \to -\frac{1}{3} R_2 \\ R_3 \to \frac{1}{6} R_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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Date :- 26-03-2020.

Show that the following transformations are linear :-

i) $T: E^3 \to E^3$ defined by
$T((x_1, x_2, x_3)) = [x_1 + x_2, x_2 + x_3, x_3 + x_1]$

sol We can show that the transformation is linear :-

let $v_1 = (x_1, x_2, x_3)$ $\quad v_2 = (y_1, y_2, y_3)$

$$
\begin{aligned}
\text{i) } T(v_1 + v_2) &= T[(x_1, x_2, x_3) + (y_1, y_2, y_3)] \\
&= T[x_1 + y_1, x_2 + y_2, x_3 + y_3] \\
&= [x_1 + y_1 + x_2 + y_2, x_2 + y_2 + x_3 + y_3, x_3 + y_3 + x_1 + y_1] \\
&= [(x_1 + x_2) + (y_1 + y_2), (x_2 + x_3) + (y_2 + y_3), (x_3 + x_1) + (y_3 + y_1)] \\
&= [x_1 + x_2, x_2 + x_3, x_3 + x_1] + [y_1 + y_2, y_2 + y_3, y_3 + y_1] \\
&= T(x_1, x_2, x_3) + T(y_1, y_2, y_3) \\
&= T(v_1) + T(v_2)
\end{aligned}
$$

$$
\begin{aligned}
\text{ii) } T(\alpha v_1) &= T[\alpha(x_1, x_2, x_3)] \\
&= T[\alpha x_1, \alpha x_2, \alpha x_3] \\
&= [\alpha x_1 + \alpha x_2, \alpha x_2 + \alpha x_3, \alpha x_3 + \alpha x_1] \\
&= [\alpha(x_1 + x_2), \alpha(x_2 + x_3), \alpha(x_3 + x_1)] \\
&= \alpha[x_1 + x_2, x_2 + x_3, x_3 + x_1] \\
&= \alpha\, T[(x_1, x_2, x_3)] \\
&= \alpha\, T(v_1)
\end{aligned}
$$

$\Rightarrow$ T is a linear transformation.

ii) $T: E^3 \to E^3$ defined by

$T((x_1, x_2, x_3)) = v \times (x_1, x_2, x_3)$ where v is a fixed vector in E^3.

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$\Rightarrow x_1 - x_2 + x_3 = 0$

let $x_2 = b \quad x_3 = c$

$x_1 = b - c$

The eigenvalues are $\lambda_2 = -2$ and the vector corresponding it are:-

$= \{(b-c, b, c) : b, c \in R\}$

$= \{b(1,1,0) + c(-1,0,1) : b, c \in R\}$

$= \{[1,1,0], [-1,0,1]\}$

Eigenvalues $\rightarrow \lambda_1 = 4 \quad \lambda_2 = -2$

Eigenvectors $\rightarrow \{[1,1,2], [1,1,0], [-1,0,1]\}$

Q2. Find the eigenvalues and the corresponding eigenspaces of the 3X3 matrix :-

$$A = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

Sol. the characteristics polynomial of matrix A is :-

$$|A - xI| = P_A(x) = \begin{bmatrix} 5-x & 4 & 2 \\ 4 & 5-x & 2 \\ 2 & 2 & 2-x \end{bmatrix}$$

$\Rightarrow (5-x)[(5-x)(2-x) - 4] - 4[4(2-x) - 4] + 2[8 - 2(5-x)]$

$\Rightarrow (5-x)(10 + x^2 - 7x - 4) - 4(8 - 4x - 4) + 2(8 - 10 + 2x)$

$\Rightarrow (5-x)(x^2 - 7x + 6) - 4(4 - 4x) + 2(2x - 2)$

$\Rightarrow 5x^2 - 35x + 30 - x^3 + 7x^2 - 6x - 16 + 16x + 4x - 4$

$\Rightarrow -x^3 + 12x^2 - 21x + 10$

$\Rightarrow -(x-1)(x^2 + 10 - 11x)$

$\Rightarrow -(x-1)(x-1)(x-10)$

$\Rightarrow -(x-10)(x-1)^2$

eigenvalues are $x = 10$ and $x = 1$ i.e. $\lambda_1 = 10 \quad \lambda_2 = 1$

vi) $T: C^2 \to C^2$ defined by
$$T((z_1, z_2)) = (z_1 + z_2, z_1 - 2z_2)$$

Sol We can show that the transformation is linear :-
Let $v_1 = (x_1, x_2)$ $v_2 = (y_1, y_2)$

i) $T(v_1 + v_2) = T[(x_1, x_2) + (y_1, y_2)]$
$= T[(x_1 + y_1, x_2 + y_2)]$
$= [x_1 + y_1 + x_2 + y_2, x_1 + y_1 - 2(x_2 + y_2)]$
$= [(x_1 + x_2) + (y_1 + y_2), (x_1 - 2x_2) + (y_1 - 2y_2)]$
$= [x_1 + x_2, x_1 - 2x_2] + [y_1 + y_2, y_1 - 2y_2]$
$= T((x_1, x_2)) + T((y_1, y_2))$
$= T(v_1) + T(v_2)$

ii) $T(\alpha v_1) = T(\alpha(x_1, x_2))$
$= T((\alpha x_1, \alpha x_2))$
$= [\alpha x_1 + \alpha x_2, \alpha x_1 - 2\alpha x_2]$
$= [\alpha(x_1 + x_2), \alpha(x_1 - 2x_2)]$
$= \alpha[x_1 + x_2, x_1 - 2x_2]$
$= \alpha T(v_1)$

$\Rightarrow$ T is a linear transformation

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3. Existence of Additive Identity :-

We must show that there is a zero function satisfying

$v_1 + v_2 = v_1$ where v_2 is the zero vector

Let $v_1 = [x_1, x_2]$, now

$v_2 = [x_{21}, x_{22}]$

$v_1 + v_2 = (x_1, x_2) + (x_{21}, x_{22}) = v_1$

$\Rightarrow (x_1 + x_{21}, x_2 + x_{22}) = (x_1, x_2)$

$\Rightarrow x_1 + x_{21} = x_1$ & $x_2 + x_{22} = x_2$

$\Rightarrow x_{21} = 0$ & $x_{22} = 0$

$\Rightarrow (x_{21}, x_{22}) = (0,0) \in R$

$\Rightarrow v_2 = (0,0)$ is the additive identity

4. Existence of Additive Inverse :-

We must show that there exists a vector '$-v_1$' such that

$v_1 = (x_1, x_2)$

$v_1 + (-v_1) = 0 = v_2$

$\Rightarrow (x_1, x_2) + (-v_1) = (x_{21}, x_{22})$

$\Rightarrow -v_1 = (0,0) - (x_1, x_2)$

$\Rightarrow -v_1 = (-x_1, -x_2) \in V$

Thus, each element of V has an additive inverse.

5. Distributivity :-

Let $a \in R$, and $v_1, v_2 \in V$, then for all $v_1 = (x_1, x_2)$

$v_2 = (y_1, y_2)$

$$\Rightarrow a(v_1 + v_2) = a[(x_1, x_2) + (y_1, y_2)]$$
$$= a[(x_1 + y_1), (x_2 + y_2)]$$
$$= [a(x_1 + y_1), a(x_2 + y_2)]$$
$$= [(ax_1 + ay_1), (ax_2 + ay_2)]$$
$$= [(ax_1, ax_2) + (ay_1, ay_2)]$$
$$= [a(x_1, x_2) + a(y_1, y_2)]$$
$$= av_1 + av_2$$

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4. Existence of Additive Inverse:
We must show that there exists a matrix '$-V_1$' such that $V_1 + (-V_1) = 0$

But, since $V_1 = [a_{ij}]_{n\times n}$ and $V_1 + (-V_1) = 0$
$[a_{ij}]_{n\times n} + [-V_1] = 0$

$\Rightarrow \boxed{-V_1 = [-a_{ij}]_{n\times n}}$

$\Rightarrow$ Because we need to take matrix with only positive entries, the additive inverse does not exist

$\Rightarrow$ V is not a vector space

Let $V = R$ and let the operations be the standard addition & multiplication in R

R is the set of all ordered 1 tuple (x_1) of real numbers. R is a vector space, with the standard operations of addition & multiplication.

Also, R denotes all the real nos. which follow all the eight vector space axioms

1. Commutative over addition:-
eg:- $1 \in R \quad 2 \in R \qquad \Rightarrow a \in R \quad b \in R$
$1 + 2 = 3 = 2 + 1 \qquad \Rightarrow a + b = b + a$
$\Rightarrow 1 + 2 = 2 + 1$

2. Associative over addition:-
eg:- $1 \in R \quad 2 \in R \quad 3 \in R \qquad \Rightarrow a \in R, b \in R \quad c \in R$
$(1 + 2) + 3 = 3 + 3 \qquad \Rightarrow (a + b) + c = a + (b + c)$
$= 6 = 1 + (2 + 3)$

3. Additive Identity :-
eg:- $1 \in R \quad 0 \in R \qquad \Rightarrow a \in R \quad 0 \in R$
$\Rightarrow 1 + 0 = 1 = 0 + 1 \qquad \Rightarrow a + 0 = a = 0 + a$

Additive Inverse :-
eg:- $1 \in R \quad -1 \in R \qquad \Rightarrow a \in R \quad -a \in R$
$1 + (-1) = 0 = (-1) + (1) \qquad \Rightarrow a + (-a) = 0 = (-a) +$

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Date :- 23.03.2020.

1. If $A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & -1 & 2 \\ -1 & -2 & 5 & 4 \end{bmatrix}$ find a basis for null spac

Null matrix can be found by finding the solution set $Ax=0$

The augumented matrix of the system is

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 3 & 4 & -1 & 2 & 0 \\ -1 & -2 & 5 & 4 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 0 & 1 & -7 & -7 & 0 \\ 0 & -1 & 7 & 7 & 0 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 + R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 9 & 10 & 0 \\ 0 & 1 & -7 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \text{Reduced row echelon form}$$

The linear system of equations are :-

$x_1 + 9x_3 + 10x_4 = 0$

$x_2 - 7x_3 - 7x_4 = 0$

Let $x_3 = b$ and $x_4 = c$

$x_1 = -9b - 10c \qquad x_2 = 7b + 7c$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9b-10c \\ 7b+7c \\ b \\ c \end{bmatrix} = b\begin{bmatrix} -9 \\ 7 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -10 \\ 7 \\ 0 \\ 1 \end{bmatrix}$$

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Q4. Let $V = \{$ ordered pairs $(x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$ with the operations $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ and $r(x_1, x_2) = (rx_1, rx_2)$.

Sol. We need to prove that V is a vector space. We have given that V satisfies closure under addition and scalar multiplication.
Now, we just need to verify the vector space axioms which are as follows:-

1. Commutative Law of Addition :-

Let $v_1 = (x_1, x_2)$ and $v_2 = (y_1, y_2)$ such that $v_1, v_2 \in V$

$$\begin{aligned} v_1 + v_2 &= (x_1, x_2) + (y_1, y_2) \\ &= (x_1 + y_1, x_2 + y_2) \\ &= (y_1 + x_1, y_2 + x_2) \\ &= (y_1, y_2) + (x_1, x_2) \\ v_1 + v_2 &= v_2 + v_1 \end{aligned}$$

$$\Rightarrow \quad v_1 + v_2 = v_2 + v_1$$

Thus, commutative law of addition holds

2. Associative Law of Addition :-

Let $v_1 = (x_1, x_2)$, $v_2 = (y_1, y_2)$ and $v_3 = (z_1, z_2)$ such that $v_1, v_2, v_3 \in V$.

$$\begin{aligned} (v_1 + v_2) + v_3 &= [(x_1, x_2) + (y_1, y_2)] + (z_1, z_2) \\ &= [(x_1 + y_1, x_2 + y_2)] + (z_1, z_2) \\ &= [(x_1 + y_1 + z_1, x_2 + y_2 + z_2)] \\ &= [x_1 + (y_1 + z_1), x_2 + (y_2 + z_2)] \\ &= (x_1, x_2) + [(y_1 + z_1), (y_2 + z_2)] \\ &= (x_1, x_2) + [(y_1, y_2) + (z_1, z_2)] \\ &= v_1 + [v_2 + v_3] \end{aligned}$$

$$(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$$

Thus, associative law of addition holds in V.

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5. Distributivity

$a \in R \quad u \in R \quad v \in R$

eg:- let $a = 4$

$\Rightarrow a(u+v)$

$\Rightarrow au + av$

$a(1+4) = 5(4) = 20$

$= a(1) + a(4)$

$= (4 \times 1) + (4)(4) = 20$

6. Distributivity

eg:- let $a = 1 \quad b = 2 \quad u = 3$

$\forall a, b, u \in R$

$(a+b)(u) = (1+2)(3)$

$= (3)(3) = 9$

$= (1)(3) + (2)(3)$

$= au + bu$

7. Associative over scalar multiplication

$\forall a, b, u \in R$

let $a = 4 \quad b = 5 \quad u = 1$

$(ab)(u) = (4 \times 5)(1) = 20$

$= a(bu) = (4)(5 \times 1) = 20$

$\Rightarrow (ab)u = a(bu)$

8. Identity under scalar multiplication

$\Rightarrow 1(a) = a \quad \forall a \in R$

$a = 2 \in R$

$1 \cdot (2) = 2 = (2)(1)$

Hence, $V = R$ is a vector space

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