(2) We have $F_1(s) = f_2(s) = \dfrac{s}{s^2+a^2} = L\{\cos at\}$.

By convolution theorem

$$L^{-1}\{f_1(s)f_2(s)\} = \int_0^t F_1(y)F_2(t-y)\,dy$$

$$\therefore L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\left(\frac{s}{s^2+a^2}\right)\left(\frac{s}{s^2+a^2}\right)\right\} = \int_0^t \cos ay$$

$$= \int_0^t \cos ay \cos a(t-y)\,dy$$

$$= \frac{1}{2}\int_0^t \left[\cos at + \cos(2ay - at)\right] dy$$

$$= \frac{1}{2}\left[y\cos at + \frac{1}{2a}\sin(2ay - at)\right]_0^t$$

$$= \frac{1}{2}\left[t\cos at + \frac{1}{a}\sin at\right] = \frac{1}{2a}\left[at\cos at + \sin at\right]$$

(3) $f_1(s) = f_2(s) = \dfrac{1}{s^2+a^2} = L\left\{\dfrac{\sin at}{a}\right\}$.

$$\therefore L^{-1}\{f_1(s)f_2(s)\} = L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \int_0^t \frac{\sin ay}{a}\cdot\frac{\sin a(t-y)}{a}\,dy$$

$$= \frac{1}{2a^2}\int_0^t \{\cos(2ay - at) - \cos at\}\,dy$$

$$= \frac{1}{2a^2}\left[\frac{\sin(2ay - at)}{2a} - y\cos at\right]_0^t$$

$$= \frac{\sin at}{2a^3} - \frac{t}{2a^2}\cos at.$$

Method of Partial Fractions for Evaluation of Inverse Laplace transform.

Note on Partial fractions:- A fraction of the form

$$\frac{a_0 s^m + a_1 s^{m-1} + \dots + a_m}{b_0 s^n + b_1 s^{n-1} + \dots + b_n}$$

in which m and n are positive integers, is called a rational algebraic fraction. When the numerator is of a lower degree than the denominator it is called a proper fraction.

To resolve a given fraction into partial fractions, we first factorise the denominator into real factors. These will be either linear or quadratic, and some factors repeated. We know that a proper fraction can be resolved into a sum of partial fractions such that.

(i) to a non-repeated linear factor $s-a$ in the denominator, corresponds a partial fraction of the form $\frac{A}{s-a}$;

(ii) to a repeated linear factor $(s-a)^r$ in the denominator, corresponds the sum of r partial fractions of the form

$$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots \frac{A_r}{(s-a)^r};$$

(iii) to a non-repeated quadratic factor s^2+as+b in the denominator, corresponds a partial fraction of the form $\frac{As+B}{s^2+as+b}$;

(iv) to a repeated quadratic factor $(s^2+as+b)^r$ in the denominator, corresponds the sum of r partial fractions of the form

$$\frac{A_1 s+B_1}{s^2+as+b} + \frac{A_2 s+B_2}{(s^2+as+b)^2} + \cdots \frac{A_r s+B_r}{(s^2+as+b)^r}$$

Then, we have to determine the unknown constants.

Inverse Transforms of some functions:-

(1) $L^{-1}\left(\frac{1}{s}\right) = 1$ (2) $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$, (3) $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$,

(4) $L^{-1}\left[\frac{1}{(s-a)^n}\right] = \frac{e^{at}\, t^{n-1}}{(n-1)!}$ (5) $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$

(6) $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$ (7) $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at$

(8) $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$ (9) $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right] = \frac{1}{b}e^{at}\sin bt$,

(10) $L^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right] = e^{at}\cos bt$. (11) $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a}t\sin t$

(12) $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}(\sin at - at\cos t)$

Example: Find the inverse transform of

(1) $\frac{s^2-3s+4}{s^3}$ (2) $\frac{s+2}{s^2-4s+13}$

(1) $L^{-1}\left(\frac{s^2-3s+4}{s^3}\right) = L^{-1}\left(\frac{1}{s}\right) - 3L^{-1}\left(\frac{1}{s^2}\right) + 4L^{-1}\left(\frac{1}{s^3}\right)$

$= 1 - 3t + 4\cdot\frac{t^2}{2!} = 1-3t+2t^2$

(2) $L^{-1}\left(\frac{s+2}{s^2-4s+13}\right) = L^{-1}\left(\frac{s+2}{(s-2)^2+9}\right) = L^{-1}\left(\frac{s-2+4}{(s-2)^2+3^2}\right)$

$= L^{-1}\left(\frac{s-2}{(s-2)^2+3^2}\right) + 4L^{-1}\left(\frac{1}{(s-2)^2+3^2}\right)$

$= e^{2t}\cos 3t + \frac{4}{3}e^{2t}\sin 3t.$

Example:- Find the inverse transforms of

(i) $\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$, (ii) $\frac{4s+5}{(s-1)^2(s+2)}$

Soln.

(1) Here the denominator is $= (s-1)(s-2)(s-3)$

So let $\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$

~~Solving~~ the. $A = \frac{2.1^2-6.1+5}{(1-2)(1-3)} = \frac{1}{2}$.

$B = \frac{2\cdot2^2-6\cdot2+5}{(2-1)(2-3)} = -1$

$C = \frac{2\cdot3^2-6\cdot3+5}{(3-1)(3-2)} = \frac{5}{2}$

$\therefore L^{-1}\left(\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right) = \frac{1}{2}L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{s-2}\right) + \frac{5}{2}L^{-1}\left(\frac{1}{s-3}\right)$

$= \frac{1}{2}e^t - e^{2t} + \frac{5}{2}e^{3t}.$

(2) let $\frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2}$

$A = \frac{1}{3}, B = 3, C = -\frac{1}{3}$.

$L^{-1}\left(\frac{4s+5}{(s-1)^2(s+2)}\right) = \frac{1}{3}e^t + 3te^t - \frac{1}{3}e^{-2t}$

Example 1. Find the inverse Laplace transform of

(i) $\dfrac{1}{(s+1)(s^2+1)}$ (ii) $\dfrac{s}{s^4+4a^4}$

(i) $\dfrac{1}{(s+1)(s^2+1)} = \dfrac{As+B}{s^2+1} + \dfrac{C}{s+1}$

$A = -B = -\frac{1}{2}, \quad C = \frac{1}{2}$

$\therefore \quad f(s) = \frac{1}{2}\left[\frac{1}{s+1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}\right]$

$\therefore \quad L^{-1}\{f(s)\} = \frac{1}{2}\left[e^{-t} - \cos t + \sin t\right]$

(2) $s^4 + 4a^4 = (s^2+2a^2)^2 - (2as)^2$

$= (s^2+2as+2a^2)(s^2-2as+2a^2)$

$\therefore$ Let $\dfrac{s}{s^4+4a^4} = \dfrac{As+B}{s^2+2as+2a^2} + \dfrac{Cs+D}{s^2-2as+2a^2}$

$A = C = 0, \quad B = -\frac{1}{4a}, \quad D = \frac{1}{4a}$

$\therefore L^{-1}\left(\frac{s}{s^4+4a^4}\right) = -\frac{1}{4a} L^{-1}\left(\frac{1}{s^2+2as+2a^2}\right) + \frac{1}{4a} L^{-1}\left(\frac{1}{s^2-2as+2a^2}\right)$

$= -\frac{1}{4a} L^{-1}\left(\frac{1}{(s+a)^2+a^2}\right] + \frac{1}{4a} L^{-1}\left(\frac{1}{(s-a)^2+a^2}\right].$

$= -\frac{1}{4a} \cdot \frac{1}{a} e^{-at} \sin at + \frac{1}{4a} \cdot \frac{1}{a} e^{at} \sin at$

$= \frac{1}{2a^2} \sin at \left(\frac{e^{at} - e^{-at}}{2}\right) = \frac{1}{2a^2} \sin at \sinh at$

Q1. Using Convolution Th. find $L^{-1}\left[\frac{1}{s^2(s+1)^2}\right]$

$f_1(t) = t, \; f_2(t) = te^{-t}$, Ans $= te^{-t} + 2e^{-t} + t - 2$.

$\dfrac{s}{s^4+4a^4}$

Applications of Laplace Transform :-

(1) Evaluation of ~~Laplace~~ Definite integrals

Since $f(s) : L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$.

Example: Evaluate the following integrals

(1) $\int_0^\infty \frac{e^{-t}\sin t}{t}\, dt$ (2) $\int_0^\infty t^2 e^{-t} \sin t\, dt = \frac{1}{2}$

Soln We have $L\left\{\frac{\sin at}{t}\right\} = \cot^{-1}\frac{s}{a}$.

$$\therefore \int_0^\infty e^{-st} \frac{\sin at}{t}\, dt = \cot^{-1}\frac{s}{a}$$

Taking $a=1$, $s=1$, we get

$$\int_0^\infty e^{-t}\frac{\sin t}{t}\, dt = \cot^{-1}(1) = \frac{\pi}{4}.$$

(2) We have $f(s) = L\{\sin t\} = \frac{1}{s^2+1}$

$$\therefore L\{t^2 \sin t\} = (-1)^2 \frac{d^2}{ds^2} L\{\sin t\}$$

$$= \frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right) = \frac{d}{ds}\left[\frac{-2s}{(s^2+1)^2}\right] = \frac{-2(1-3s^2)}{(1+s^2)^3}$$

$$= \left[\frac{8s^2}{(s^2+1)^3} - \frac{2}{(s^2+1)^2}\right]$$

(2) Solution of Differential eqns with constant coefficients

Consider the linear differential equation

$$a_0 \frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_2 y = F(t) \quad \text{—(1)}$$

with constant coeff a_0, a_1 and a_2.

Taking Laplace transform of both the sides, we get

$$a_0 L\left\{\frac{d^2y}{dt^2}\right\} + a_1 L\left\{\frac{dy}{dt}\right\} + a_2 L\{y\} = L\{F(t)\} \quad \text{—(2)}$$

But from the property of Laplace transform of derivative

$L[F'(t)] = sf(s) - F(0)$

$L(F''(t)) = s^2 f(s) - sF(0) - F'(0)$

Here $F(t) = y$

$$L\left\{\frac{dy}{dt}\right\} = sL\{y\} - [y]_{t=0}$$

$$L\left\{\frac{d^2y}{dt^2}\right\} = s^2 L\{y\} - s[y]_{t=0} - \left(\frac{dy}{dt}\right)_{t=0}$$

and $L\{F(t)\} = f(s)$

Then eq ② gives

$$a_0\left[s^2 L\{y\} - s(y)_{t=0} - \left(\frac{dy}{dt}\right)_{t=0}\right] + a_1\left[sL\{y\} - (y)_{t=0}\right] + a_2 L\{y\} = f(s)$$

or $$(a_0 s^2 + a_1 s + a_2) L\{y\} = f(s) + (a_0 s + a_1)(y)_{t=0} + a_0\left(\frac{dy}{dt}\right)_{t=0}$$

or $$L(y) = \frac{f(s) + (a_0 s + a_1)(y)_{t=0} + a_1\left(\frac{dy}{dt}\right)_{t=0}}{a_0 s^2 + a_1 s + a_2} \quad -③$$

The R.H.S of the eqn is of the form $\frac{G(s)}{H(s)}$, therefore the solution of diff eqn ① becomes

$$y(t) = L^{-1}\left\{\frac{G(s)}{H(s)}\right\}.$$

<u>Example:</u>– Using Laplace transformation method solve the diff eqn $y'' + 9y = 0$, satisfying the initial conditions

$$y(0) = 0 \text{ and } y'(0) = 2$$

Given that $L^{-1}\left\{\frac{3}{s^2+9}\right\} = \sin 3t$.

<u>Soln</u>

$$y'' + 9y = 0$$

Taking Laplace transform, we get

$$L\{y''\} + 9L\{y\} = L\{0\}.$$

ie $$s^2 L\{y\} - s(y)_{t=0} - \left(\frac{dy}{dt}\right)_{t=0} + 9L\{y\} = L\{0\}$$

ie $$(s^2+9) L\{y\} = L(0) + s(y)_{t=0} + \left(\frac{dy}{dt}\right)_{t=0}$$

$$= L(0) + s y(0) + y'(0)$$

$$= 0 + s \cdot 0 + 2 = 2.$$

$\frac{As+B}{s-3i} + \frac{C}{s+3i}$

ie $$L\{y\} = \frac{2}{s^2+9} = \frac{2}{3}\left(\frac{3}{s^2+9}\right)$$

$\frac{s+1}{(s-3i)(s+3i)}$

Taking inverse Laplace transform, we get

$$y(t) = \frac{2}{3} L^{-1}\left\{\frac{3}{s^2+9}\right\} = \frac{2}{3}\sin 3t.$$

$\frac{1}{s-1}$

Example:–

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 2x = 0 \quad ; \quad x_0 = x_1 = 1$$

$L\{x\} = \frac{s-1}{s^2-2s+2}$, $x(t) = L^{-1}\left\{\frac{s-1}{s^2-2s+2}\right\} = e^t \cos t$.

Q.1

$$(s+2)^2 L[y] = \frac{2}{s^3} + \frac{1}{2}$$

$$L[y] = \frac{4+s^3}{2s^3(s+2)^2}$$

$$\frac{s^3+4}{s^3(s+2)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+2} + \frac{E}{(s+2)^2}$$

$A = \frac{3}{4}$

$B = -1$

$C = 1$

$D = -\frac{3}{4}$

$E = \frac{1}{2}$

$L[t] = \frac{1}{s^2}$

$L[t^2] = \frac{2!}{s^3}$

$$L^{-1}\left[\frac{1}{s}\right] = 1 \quad L^{-1}\left[\frac{1}{s^2}\right] = t, \quad L^{-1}\left[\frac{1}{s^3}\right] = \frac{t^2}{2!}$$

$$L^{-1}\left[\frac{1}{s+2}\right] = e^{-2t} \quad L^{-1}\left[\frac{1}{(s+2)^2}\right] = te^{-2t}$$

Ans. $Y(t) = \frac{1}{2}\left[\frac{3}{4} - t + \frac{t^2}{2} - \frac{3}{4}e^{-2t} + \frac{1}{2}te^{-2t}\right]$

Q.2 $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 4 \qquad Y(0) = 1, \ Y'(0) = 0$

$$L[y] = \frac{s^2-3s+4}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$$

$A = 2, \ B = -2, \ C = 1$

$$y = 2L^{-1}\left[\frac{1}{s}\right] - 2L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{1}{s-2}\right]$$

Ans $y = 2 - 2e^{t} + e^{2t}$

Q.3 $\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 1 - 2t \qquad Y(0) = 0, \ Y'(0) = 4$

$$L[y] = \frac{4s^2+s-2}{s^2(s-1)(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s+2}$$

$A = 0, \ B = 1, \ C = 1, \ D = -1$

$$y = L^{-1}\left[\frac{1}{s^2}\right] + L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right] = t + e^{t} - e^{-2t}$$

Solve $(D^2-3D+2)y = 1-e^{2t}$

when $y(0)=1$

and $Dy(0)=0$.

Soln. $\frac{d^2}{}$ $y''(t) - 3y'(t) + 2y = 1-e^{2t}$.

Take Laplace Tranform

$L\{y''(t)\} - 3L\{y'(t)\} + 2L\{y(t)\} = L\{1\} - L\{e^{2t}\}$.

or $p^2L(y) - py(0) - y'(0) - 3(pL(y) - y(0)]$

$+2L(y) = \frac{1}{p} - \frac{1}{p-2}$

or $(p^2-3p+2)L(y) - p + 3 = \frac{-2}{p(p-2)}$

or $(p-1)(p-2)L(y) = p-3 - \frac{2}{p(p-2)}$

or $L(y) = \frac{p^3-5p^2+6p-2}{p(p-1)(p-2)^2} = \frac{(p-1)(p^2-4p+2)}{p(p-2)^2(p-1)}$

$= \frac{p^2-4p+2}{p(p-2)^2}$

$= \frac{1}{2p} + \frac{1}{2(p-2)} - \frac{1}{(p-2)^2}$

Taking Inverse transform

$y = \frac{1}{2} + \frac{1}{2}e^{2t} - te^{2t}$.

Solution of Simultaneous ordinary differential equations

Solve $(D^2-1)x + 5Dy = t$.

$-2Dx + (D^2-4)y = -2$

if $x = 0 = Dx = Dy$ when $t=0$

Taking Laplace transforms of both sides of the given equations, we get

$L\{x''(t)\} - L\{x(t)\} + 5L\{y'(t)\} = L\{t\}$.

and $-2L\{x'(t)\} + L\{y''(t)\} - 4L\{y(t)\} = -2L\{1\}$

or $p^2 L\{x(t)\} - p\,x(0) - x'(0) - L\{x(t)\} + 5\left[p L\{y(t)\} - y(0)\right] = 1/p^2$

and $-2\left[p L\{x(t)\} - x(0)\right] + p^2 L\{y(t)\} - p\,y(0) - y'(0) - 4 L\{y(t)\} = -2/p$

or $(p^2-1) L\{x(t)\} + 5p\, L\{y(t)\} = \frac{1}{p^2}$

and $-2p\, L\{x(t)\} + (p^2-4) L\{y(t)\} = -\frac{2}{p}$

Solving these two algebraic equations for $L\{x(t)\}$ and $L\{y(t)\}$,

$$L\{x(t)\} = \frac{-11p^2 - 4}{p(p^2+1)(p^2+4)} = -\frac{1}{p^2} + \frac{5}{p^2+1} - \frac{4}{p^2+4} \quad -(1)$$

$$\text{and } L\{y(t)\} = \frac{-2p^2+4}{p(p^2+1)(p^2+4)} = \frac{1}{p} - \frac{2p}{p^2+1} + \frac{p}{p^2+4} \quad -(2)$$

Taking inverse transforms.

$x(t) = -t + 5\sin t - 2\sin 2t$

$y(t) = 1 - 2\cos t + \cos 2t$.

Solve.

$$\frac{dx}{dt} + x + y = 0 \qquad (x(0) = 1,\ y(0) = 1)$$

$$\frac{dy}{dt} + y + x = 0$$

Ans $x = y = e^{-2t}$

$L[\frac{dx}{dt}] + L[x] + L[y] = 0$

$L[\frac{dy}{dt}] + L[y] + L[x] = 0$

$Dx + x + y = 0$ $\quad D^2x + 2Dx = 0$

$Dy + y + x = 0$ $\quad D(D+2) = 0$

$Dx = Dy$ $\quad D = 0,\ D = -2$.

$x = y$.

$D^2x + Dx + Dy = 0$

$D^2x + Dx - y - x = 0$

$D^2x + Dx - 2x = 0$

$\lambda^2 + \lambda - 2 = 0$

$L^{-1}[f'(p)] = e^{3t} - e^{-2t}$

$(-1)^n L[tF(t)] = (-1)^n \frac{d}{ds} L(F(t)) = (-1)^n \frac{df(s)}{ds}$

$(-1)^n L[t e^{3t} - t e^{-2t}]$

$L[tF(t)] = -\frac{df(s)}{ds}$

$tF(t) = -L^{-1}[f'(s)]$

$t L^{-1}[f(s)] = -L^{-1}[f'(s)]$

Solution of simultaneous differential eqns by Laplace transforms.

eg. 1 $\frac{dx}{dt} + y = 0$ and $\frac{dy}{dt} - x = 0$ $\quad x(0) = 1,\ y(0) = 0$

$L\left[\frac{dx}{dt}\right] + L[y] = 0 \quad \Rightarrow \quad sL[x] - x(0) + L[y] = 0$

$L\left[\frac{dy}{dt}\right] - L[x] = 0 \quad\quad sL[y] - y(0) - L[x] = 0$

$sL[x] - 1 + L[y] = 0 \quad -①$

$sL[y] - L[x] = 0 \quad -②$

② × s + ①

$s^2 L[y] + L[y] - 1 = 0$

$L[y] = \frac{1}{s^2+1}$, $y(t) = \sin t$.

From ② $L[x] = s\,L[y] = \frac{s}{s^2+1} \Rightarrow x(t) = \cos t$.

eg. 2 $\cancel{(D+1)}$ $\frac{dx}{dt} - y = e^t$, $\frac{dy}{dt} + x = \sin t$ $\quad x(0) = 1,\ y(0) = 0$

$L\left[\frac{dx}{dt}\right] - L[y] = L[e^t]$

$L\left[\frac{dy}{dt}\right] + L[x] = L[\sin t]$

$sL(x) - x(0) - L[y] = \frac{1}{s-1}$

$sL(y) - y(0) + L[x] = \frac{1}{s^2+1}$

$sL(x) - 1 - L[y] = \frac{1}{s-1} \quad -①$

$sL(y) + L[x] = \frac{1}{s^2+1} \quad -②$

① × s + ②

$s^2 L(x) - s - \cancel{sL[y]} = \frac{s}{s-1} + \frac{1}{s^2+1}$

$+ \cancel{sL(y)} + L[x]$

$s^2 L(x) + L(x) = \frac{s}{s-1} + \frac{1}{s^2+1} + s = \frac{s(s^2+1) + s - 1 + (s^2-s)(s^2+1)}{(s-1)(s^2+1)}$

$L(x) = \frac{s^4 + s^2 + s - 1}{(s-1)(s^2+1)^2} = \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{s+1}{s^2+1} + \frac{1}{(s^2+1)^2}$

$L(y) = \frac{-s^3 + s^2 - 2s}{(s-1)(s^2+1)^2} = -\frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{s-1}{s^2+1} + \frac{s}{(s^2+1)^2}$

$\Rightarrow x(t) = \frac{1}{2}e^t + \frac{1}{2}\cos t + \frac{1}{2}\sin t + \frac{1}{2}(\sin t - t\cos t) = \frac{1}{2}[e^t + \cos t + 2\sin t - t\cos t]$

$y(t) = -\frac{1}{2}e^t + \frac{1}{2}\cos t - \frac{1}{2}\sin t + \frac{1}{2}t\sin t = \frac{1}{2}[-e^t - \sin t + \cos t + t\sin t]$