Date - 1/04/2020

Lecture - 19

{ Basis for a vector space }

Defn Basis : Let V be a vector space, and let S be subset of V then S is called a basis for V if it is linearly Indep and spans i.e

span(S) = V and S is linearly independent set

Q Show that $S = \{[1,0 \ldots 0], [0,1,0 \ldots] \ldots [0,0 \ldots 1]\}$ is basis for the vector space $\mathbb{R}^n$.

Soln Let $e_1 = [1,0,0 \ldots 0]$

$e_2 = (0,1,0 \ldots 0)$

$e_3 = (0,0,1,0 \ldots 0)$

$\vdots$

$e_n = (0,0 \ldots 0,1)$

To show S is linearly Indep

Let $a_1, a_2 \ldots a_n \in \mathbb{R}$ st

$a_1 e_1 + a_2 e_2 \ldots + a_n e_n = 0$

$\Rightarrow a_1(1,0 \ldots 0) + a_2(0,1,0 \ldots 0) \ldots + a_n(0,0 \ldots 1) = 0 = (0,0 \ldots 0)$

$\Rightarrow (a_1, 0, 0 \ldots 0) + (0, a_2, 0 \ldots 0) + \ldots + (0,0 \ldots a_n) = (0,0 \ldots 0)$

$\Rightarrow (a_1, a_2 \ldots a_n) = (0,0 \ldots 0)$

$\Rightarrow a_1 = 0, a_2 = 0 \ldots = a_n = 0$

$\Rightarrow$ By defn of linearly Independent set

$\Rightarrow S = \{e_1, e_2 \ldots e_n\}$ is linearly Independent

Next To show $\text{span}(S) = \mathbb{R}^n$

$\because S = \{(1,0,0 \ldots 0), (0,1,0 \ldots 0), \ldots (0,0 \ldots 1)\}$ form matrix A

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 1 \end{bmatrix}_{n \times n} = A$$

Already in reduced echlon from

$\because$ each column has leading entry (i.e pivot)

$\Rightarrow \text{span}(S) = \{a_1(1,0 \ldots 0) + a_2(0,1 \ldots 0) \ldots + a_n(0, \ldots 0,1) : a_i \in \mathbb{R}\}$

$= \{(a_1, a_2 \ldots a_n) : a_i \in \mathbb{R}\}$

$\Rightarrow \text{span}(S) = \mathbb{R}^n$

$\Rightarrow$ S is basis for $V = \mathbb{R}^n$

(121)

Qn $S = \{1, x, x^2, x^3\}$ is basis for $P_3(R)$

Soln for proving S is basis for $P_3(R)$

we shall prove two condits

(1) S is linearly Independent

(2) $span(S) = P_3(R)$

Let a_1, a_2, a_3, a_4 be scalars s.t

$a_1(1) + a_2(x) + a_3(x^2) + a_4 x^3 = 0$

$\Rightarrow a_1 + a_2 x + a_3 x^2 + a_4 x^3 = 0$

$= 0 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$

equating the coeff of x, x^2, x^3 and constant

we get

$a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0$

$\Rightarrow$ By defn of linearly Indep set

S is Linearly Independent

Next to show $span(S) = P_3(R)$

form matrix A from set $S = \{1, x, x^2, x^3\}$

(122)

$1 \to 1 + 0 x + 0 x^2 + 0 x^3$

$x \to 0 + 1 \cdot x + 0 \cdot x^2 + 0 x^3$

$x^2 \to 0 + 0 x + 1 \cdot x^2 + 0 \cdot x^3$

$x^3 \to 0 + 0 \cdot x + 0 x^2 + 1 \cdot x^3$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

matrix A is already reduced row echlon from

$\Rightarrow span(S) = \{a_0(1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3) + a_1(0 + 1 \cdot x + 0 \cdot x^2 + 0 x^3) + a_2(0 + 0 x + 1 \cdot x^2 + 0 \cdot x^3) + a_3(0 + 0 \cdot x + 0 x^2 + 1 \cdot x^3) : a_0, a_1, a_2, a_3 \in R\}$

$= \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 : a_0, a_1, a_2, a_3 \in R\}$

$\Rightarrow span(S) = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 : a_0, a_1, a_2, a_3 \in R\}$

$\Rightarrow span(S) = P_3(R)$

$\Rightarrow$ S is linearly independent and $span(S) = P_3(R)$

$\Rightarrow$ S forms bases for $V = P_3(R)$

(123)

Theorem $S = \{v_1, v_2, \dots, v_n\}$ of vectors in V is a basis for V if and only if every $v \in V$ can be written uniquely in the form
$$v = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$$
where $a_1, a_2, \dots, a_n \in \mathbb{R}$

Proof Suppose S is a basis for V
$\Rightarrow$ S is linearly independent and $\text{span}(S) = V$

Let $v \in V$. Since S spans V
$\Rightarrow$ there exist real numbers $a_1, a_2, \dots, a_n$ such that
$$v = a_1 v_1 + a_2 v_2 + \dots + a_n v_n \quad (1)$$

To show this representation in (1) is unique.
Suppose $b_1, b_2, \dots, b_n$ are scalars such that we have
$$v = b_1 v_1 + b_2 v_2 + \dots + b_n v_n \quad (2)$$

From (1) - (2) we get

$$v - v = (a_1 v_1 + a_2 v_2 + \dots + a_n v_n) - (b_1 v_1 + b_2 v_2 + \dots + b_n v_n)$$

(124)

(S is L I set)

$$\Rightarrow (a_1 - b_1) v_1 + (a_2 - b_2) v_2 + \dots + (a_n - b_n) v_n = 0 = 0 \cdot v_1 + 0 \cdot v_2 + \dots + 0 \cdot v_n$$

On equating we get
$$a_1 - b_1 = 0, \; a_2 - b_2 = 0, \dots, a_n - b_n = 0$$
$$\Rightarrow a_1 = b_1, \; a_2 = b_2, \dots, a_n = b_n$$

This shows that representation in (1) is unique.

Conversely for each $v \in V$ can be written uniquely in the form
$$v = v_1 a_1 + v_2 a_2 + \dots + v_n a_n \quad (A)$$

$\Rightarrow$ from (A) we can say that for each $v \in V$ $\exists$ some $v_1, v_2, \dots, v_n$ in S such that

$$v = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$$
$\Rightarrow$ S spans V i.e. $\text{span}(S) = V$

Next to show S is L.I.
Suppose $a_1, a_2, \dots, a_n \in \mathbb{R}$ such that
$$a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0 \quad (B)$$

but $0 = 0 \cdot v_1 + 0 \cdot v_2 + \dots$ each

(2)

from (3) & (4) we get

$a_1v_1 + a_2v_2 + \dots + a_nv_n = 0.v_1 + 0.v_2 + \dots + 0.v_n$

On equating
$a_1 = 0,\ a_2 = 0\ \dots\ a_n = 0$

$\Rightarrow$ S is Linearly Independent

Q. Prove that
$S = \{(1,3,-1),(2,7,-3),(4,8,-7)\}$
Spans $\mathbb{R}^3$. Examine whether S forms a basis for $\mathbb{R}^3$?
[DU (G.E.) - 2018]

Sol^n form matrix A from ... S

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 7 & -3 \\ 4 & 8 & -7 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & -4 & -3 \end{bmatrix}$$

$R_1 \to R_1 - 3R_2$
$R_3 \to R_3 + 4R_2$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & -7 \end{bmatrix} \xrightarrow{R_3 \to -\frac{1}{7}R_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - 2R_3$
$R_2 \to R_2 + R_3$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ this is reduced row echelon form

$\Rightarrow$ Span(S) is the linear combination of row of B

$\Rightarrow$ Span(S) $= \{a(1,0,0) + b(0,1,0) + c(0,0,1) : a,b,c \in \mathbb{R}\}$

$= \{(a,0,0) + (0,b,0) + (0,0,c) : a,b,c \in \mathbb{R}\}$

$= \{(a,b,c) : c \in \mathbb{R}\} = \mathbb{R}^3$

$\Rightarrow$ Span(S) $= \mathbb{R}^3 = \mathbb{R}^3$

$\rightarrow$ next to show S is Linearly Independent

$$|A| = \begin{vmatrix} 1 & 3 & -1 \\ 2 & 7 & -3 \\ 4 & 8 & -7 \end{vmatrix}$$

$= 1(-49+24) - 3(-14+12) - 1(18-28)$

$= -25 + 6 + 10 = -9 \neq 0$

$\because |A| \neq 0 \Rightarrow$ S is L.I.

∴ S is linearly Independent
and Span(S) = $\mathbb{R}^3$
⇒ S form basis for $\mathbb{R}^3$

Qn1

S = {(1, 4, 2, 0), (0, 7, 1, 0),
(-3, 1, 1, 0), (5, -2, 0, -3)}
to show S is linearly Indep
and Span(S) = $\mathbb{R}^4$

[Do your self]

Qn2 S = {(1, 2, -4, 18), (0, 3, 4, -4)
(1, 5, 2, 0), (0, 0, 3, 2)}
S is linearly Independent or
not & find Span(S) = ?

[Do your self]

Qn3 S = {(1, 0, 0, 9), (-2, 3, 2, 4)
(2, 2, 5, -5)}
Determine S is basis for
$\mathbb{R}^4$ or not

[Do your self]

Qn4 S = {[1, 0, 0, 9], [-2, 3, 2, 4]
, [2, 2, 5, -5], [5, 1, 1, 1], [7, 5, 0, 3]}
Determine S is basis for
$\mathbb{R}^4$ or not

[Do your self]

Qn5 Prove that the following
set is a basis for M_{22} by
showing that it spans M_{22} and
linearly independent

$$S = \left\{ \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 2 & -2 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix} \right\}$$

Imp
Dimension is the number of elements in any basis for V

Defn finite-dimensional vector space
A vector space V is said to be finite-dimensional if it has basis B containing a finite number of elements. The dimension of a finite-dimensional vector space V, denoted by dim V, is the number of elements in any basis for V.

A vector space V is called infinite-dimensional if it not finite dimensional. i.e V is infinite dimensional if it has no finite basis.

Result: Let V be a finite dim vector space

(1) Let S is finite subset of V & $span(S) = V$ then $\dim V \leq |S|$ moreover $|S| = \dim V$ iff S is basis for V

(b) Suppose T is linearly independent subset of V. Then T is finite and $|T| \le \dim V$. Moreover $|T| = \dim V$ if and only if T is a basis for V.

\# $|S|$ = this is cardinality of S [i.e. no of elements in S]
$\underbrace{\text{elements}}_{\text{distinct}}$

Ex $S = \{1, 2, 5, 4, 6\}$

then $|S| = 6$

Qn Examine whether the subset $S = \{[7,1,2,0], [8,0,1,-1], [1,0,0,-2]\}$ of $\mathbb{R}^4$ ~~be~~ form a basis for $\mathbb{R}^4$.

Soln form ~~matrix~~ a matrix A by using S

$$A = \begin{bmatrix} 7 & 1 & 2 & 0 \\ 8 & 0 & 1 & -1 \\ 1 & 0 & 0 & -2 \end{bmatrix}$$

[Do yourself]

Qn $S = \{[2,1], [5,1]\}$ subset of $\mathbb{R}^2$. Form basis or not.

Soln $A = \begin{bmatrix} 2 & 1 \\ 5 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1/2} \sim \begin{bmatrix} 1 & 1/2 \\ 5 & 1 \end{bmatrix}$

$R_2 \to R_2 - 5R_1$ $\qquad$ $R_2 \to -2/3 R_2$

$\begin{bmatrix} 1 & 1/2 \\ 0 & -3/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$

$R_1 \to 2R_1 - R_2$ $\qquad$ $R_1 \to R_1/2$

$\sim \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = C$

$\text{span}(S) = \{a(1,0) + b(0,1) : a, b \in \mathbb{R}\}$
$= \{(a,b) : a, b \in \mathbb{R}\}$

$\therefore$ S is L.I $\Rightarrow$ S form basis for $\mathbb{R}^2$

OR

$\therefore$ S is linearly independent & $\dim(S) = 2 = \dim(\mathbb{R}^2)$
$\Rightarrow$ S form basis for $\mathbb{R}^2$

Q) Show that the set $B = \{(1,2,1), (2,1,1), (-1,3,-5)\}$ is basis for R^3

Sol^n Form matrix A from set B

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ -1 & 3 & -5 \end{bmatrix}$$

$$|A| = 1(-5-3) - 2(-5+1) + 1(4+3)$$
$$= -11 + 10 + 7 = 6 \neq 0$$

$\Rightarrow$ Set B is linearly independent

$\Rightarrow \dim(B) = 3$ & $\dim(R^3) = 3$

$\Rightarrow \dim(B) = \dim(R^3) = 3$

Since we know that: V is finite dim vector space & S is any linearly independent subset of V then $|S| = \dim(V)$ iff S is basis for V

$\Rightarrow S = \{(1,2,1), (2,1,1), (-1,3,-5)\}$ is form basis for R^3

Q) Prove that the set $S = \{(3,1,-1), (5,2,-2), (2,2,-1)\}$ is linearly independent. Examine whether S form a basis for R^3

[DU DSE 1 2017]

Sol^n method 2

Form matrix A from S

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & -2 \\ 2 & 2 & -1 \end{bmatrix} \sim \overset{R_1 \leftrightarrow R_2}{\begin{bmatrix} 1 & 2 & 2 \\ 3 & 5 & 2 \\ -1 & -2 & -1 \end{bmatrix}}$$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to (-1)R_2$

$$\sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 + 6R_3$
$R_2 \to R_2 - 4R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

13

∵ there is a pivot in every column of the reduced row echelon form of the matrix A

⇒ span(S) = { a(1,0,0) + b(0,1,0) + c(0,0,1) : a,b,c ∈ $\mathbb{R}$ }

= { (a,b,c) : a,b,c ∈ $\mathbb{R}^3$ }

⇒ span(S) = $\mathbb{R}^3$

⇒ ~~...~~ =
∵ |A| ≠ 0 ⇒ S is linearly independent

⇒ S is form bases for V if
1. S is linearly independent
2. span(S) = V

⇒ S form bases for $\mathbb{R}^3$

Method 2. form matrix A form set S

$$A = \begin{bmatrix} 3 & 5 & 2 \\ 1 & 2 & 2 \\ -1 & -2 & +1 \end{bmatrix}$$

$|A| = 3(-2+4) - 5(-1-2) + 2(-2+2)$
$= 3 \cdot 2 + 15 \neq 0$

15

⇒ set S is linearly independent
⇒ dim(S) = 3
∵ dim($\mathbb{R}^3$) = 3

⇒ dim(S) = dim($\mathbb{R}^3$) = 3

We know if V is finite vector space & S is finite linearly independent set then |S| = dim V then S is basis for V

⇒ By above statement S is form ~~basis of~~ basis for V

Q. Show that the set S
S = {(-1,2,-3), (3,1,4), (2,-1,6)}
is a basis for $\mathbb{R}^3$
[DU (SE-2 2014)]

Sol^n form matrix A by using S

$R_1 \to (-1) R_1$

$$A = \begin{pmatrix} -1 & 2 & -3 \\ 3 & 1 & 4 \\ 2 & -1 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 3 \\ 3 & 1 & 4 \\ 2 & -1 & 6 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$ $\qquad$ $R_2 \to \frac{1}{7} R_2$

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 7 & -5 \\ 0 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -5/7 \\ 0 & 3 & 0 \end{bmatrix}$$

$R_1 \to R_1 + 2R_2$ $\qquad$ $R_1 \to R_1 - \frac{11}{7} R_3$

$R_3 \to R_3 - 3R_2$ $\qquad$ $R_3 \to R_3(7/15)$ $\qquad$ $R_2 \to R_2 + \frac{5}{7} R_3$

$$\begin{bmatrix} 1 & 0 & 11/7 \\ 0 & 1 & -5/7 \\ 0 & 0 & 15/7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 11/7 \\ 0 & 1 & -5/7 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since pivot in each column

Now

$\text{span}(S) = \{ a(1,0,0) + b(0,1,0) + c(0,0,1) : a,b,c \in \mathbb{R} \}$

$= \{ (a,b,c) : a,b,c \in \mathbb{R} \} = \mathbb{R}^3$

$\Rightarrow \text{span}(S) = \mathbb{R}^3$

$\because |S| = 3 = \dim(\mathbb{R}^3)$

$\Rightarrow$ S is basis for $\mathbb{R}^3$

Maximal linearly independent sets and minimal spanning sets

Def^n A subset S of a vector space V is said to be maximal linearly independent subset of V if S is linearly independent and any subset T of V that properly contains S is linearly dependent.

A subset S of a vector space V is said to be minimal spanning subset of V if S spans V and no proper subset of S spans V.

Qn (1) $B = \{ (2,3,0,-1), (-1,1,1,-1) \}$ and $S = \{ (1,4,1,-2), (-1,1,1,-1), (3,2,-1,0), (2,3,0,-1) \}$

(a) Show that B is maximal linearly independent subset of S.

(b) Calculate dim(Span(S)).

(c) Does Span(S) $= \mathbb{R}^4$? Why or why not? [DU CBE-2, 2018]

Q. (a) Set $B = \{(2,3,0,-1), (-1,1,1,-1)\}$

form matrix A

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \sim \xrightarrow{R_4 \leftrightarrow R_1} \begin{bmatrix} -1 & -1 \\ 3 & 1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix} \sim \xrightarrow{R_1 \to (-1)R_1} \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 - 2R_1} \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 0 & 1 \\ 0 & -3 \end{bmatrix} \sim \xrightarrow{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 1 \\ 0 & -3 \end{bmatrix} \sim \xrightarrow{R_4 \to R_4 + \dots} \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to 2R_3 + R_2} \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \sim \xrightarrow{R_2 \to R_2/(-2)} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

⇒ each column has pivot

⇒ set is linearly independent

Next to show B is maximal linearly independent subset of S for this to show any subset of S that properly contains B is linearly dependent

only subsets of S that properly contain B are

$T_1 = \{(2,3,0,-1), (-1,1,1,-1), (1,4,1,-2)\}$

$T_2 = \{(2,3,0,-1), (-1,1,1,-1), (3,2,-1,0)\}$

$T_3 = \{(2,3,0,-1), (-1,1,1,-1), (1,4,1,-2), (3,2,-1,0)\} = S$

To show T_1 is linearly dependent

$T_1 = \{(2,3,0,-1), (-1,1,1,-1), (1,4,1,-2)\}$

form a matrix A form T_1

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 4 & 1 & 3 \\ 1 & 1 & 0 \\ -2 & -1 & -1 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 + 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 5 & -5 \\ 0 & 2 & -2 \\ 0 & -3 & 3 \end{bmatrix}$$

$R_2 \to R_2/5$
$R_3 \to R_3/2$
$R_4 \to R_4/3$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$
$R_4 \to R_4 + R_2$

$R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = B \text{ (say)}$$

$\because$ 3rd column has no leading entry

$\Rightarrow$ T_1 is linearly dependent

$T_2 = \{(2,3,0,-1), (-1,1,1,-1), (3,2,-1,0)\}$

Similarly show T_2 is linearly dependent & T_3 is also linearly dependent

$\Rightarrow$ B is the maximal linearly independent subset of S

(b) dim(span(S)) = ?

Form matrix A from set S

$$A = \begin{bmatrix} 1 & 4 & 1 & -2 \\ -1 & 1 & 1 & -1 \\ 3 & 2 & -1 & 0 \\ 2 & 3 & 0 & -1 \end{bmatrix}$$

Apply elementary row operation

$$\begin{bmatrix} 1 & 4 & 1 & -2 \\ -1 & 1 & 1 & -1 \\ 3 & 2 & -1 & 0 \\ 2 & 3 & 0 & -1 \end{bmatrix} \xrightarrow[R_4 \to R_4 - 2R_1]{R_2 \to R_2 + R_1,\ R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & 4 & 1 & -2 \\ 0 & 5 & 2 & -3 \\ 0 & -10 & -4 & 6 \\ 0 & -5 & -2 & 3 \end{bmatrix}$$

$R_2 \to R_2/5$

$$\begin{bmatrix} 1 & 4 & 1 & -2 \\ 0 & 1 & 2/5 & -3/5 \\ 0 & -10 & -4 & 6 \\ 0 & -5 & -2 & 3 \end{bmatrix} \xrightarrow[R_4 \to R_4 + 5R_2]{R_3 \to R_3 + 10R_2} \begin{bmatrix} 1 & 4 & 1 & -2 \\ 0 & 1 & 2/5 & -3/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 - 4R_2$

$$\begin{bmatrix} 1 & 0 & -4/5 & 2/5 \\ 0 & 1 & 2/5 & -3/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = C \text{ (say)}$$

matrix C is the reduced row echelon form of A and

$\text{span}(S) = \{a(1, 0, -3/5, 2/5) + b(0, 1, 2/5, -3/5) : a, b \in \mathbb{R}\}$

$\dim(\text{span}(S)) = 2$ [number of L.I vector]

(c) $\text{span}(S) \neq \mathbb{R}^4$

because $\dim(\text{span}) = 2 \neq \dim \mathbb{R}^4$

Let S be subset of a vector space V then the following statements are equivalent.

(a) S is a basis for V

(b) S is minimal spanning subset for V

(c) S is maximal L.I subset of V

Qn Find a basis and the dimension for the subspace W of $\mathbb{R}^3$ spanned by the set

$S = \{(3,2,1), (1,2,0), (-1,2,-1)\}$

[DU GE-2 2016]

Soln form matrix A from set S

$R_1 \leftrightarrow R_2$

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 0 \\ -1 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & 1 \\ -1 & 2 & -1 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 + R_1$

$R_2 \to -1/4 R_2$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -4 & 1 \\ 0 & 4 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1/4 \\ 0 & 4 & -1 \end{bmatrix}$$

$R_1 \to R_1 - 2R_2$
$R_3 \to R_3 - 4R_2$

$$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix} = C \text{ (say)}$$

matrix C is the reduced row echelon form of A

Now

$\text{span}(S) = \{a(1,0,\tfrac{1}{2}) + b(0,1,-\tfrac{1}{4}) : a,b \in \mathbb{R}\}$

$\dim(\text{span}(S)) = 2$

$\because \text{span}(S) = \{a\underline{(2,0,1)} + b\underline{(0,4,-1)} : a,b \in \mathbb{R}\}$

$(2,0,1)$ & $(0,4,-1)$ LI

$\Rightarrow B = \{(2,0,1), (0,4,-1)\}$ form basis for $\text{span}(S) = W$

Q. Find a basis and dimension for the subspace W of $\mathbb{R}^3$ defined by

$W = \{(x,y,z) \in \mathbb{R}^3 : 2x-3y+z=0\}$

[DU GE-2 2017]

Sol. $2x-3y+z=0$

$x = \frac{3}{2}y - \frac{z}{2}$

put $y=b$ & $z=c$

$\Rightarrow x = \frac{3}{2}b - \frac{1}{2}c$

$W = \{[\frac{3}{2}b - \frac{1}{2}c, b, c] : b,c \in \mathbb{R}\}$

$= \{[\frac{3}{2}b, b, 0] + [-\frac{c}{2}, 0, c] : b,c \in \mathbb{R}\}$

$= \{b[\frac{3}{2}, 1, 0] + c[-\frac{1}{2}, 0, 1] : b,c \in \mathbb{R}\}$

$= \text{span}\{(\frac{3}{2},1,0), (-\frac{1}{2},0,1)\}$

$= \text{span}\{(3,2,0), (-1,0,2)\}$

$\Rightarrow$ set $S = \{(3,2,0), (-1,0,2)\}$ spans the subspace W

$\because (3,2,0), (-1,0,2)$ LI

$\Rightarrow S$ is basis for W

$\Rightarrow \dim W = 2 = |S|$