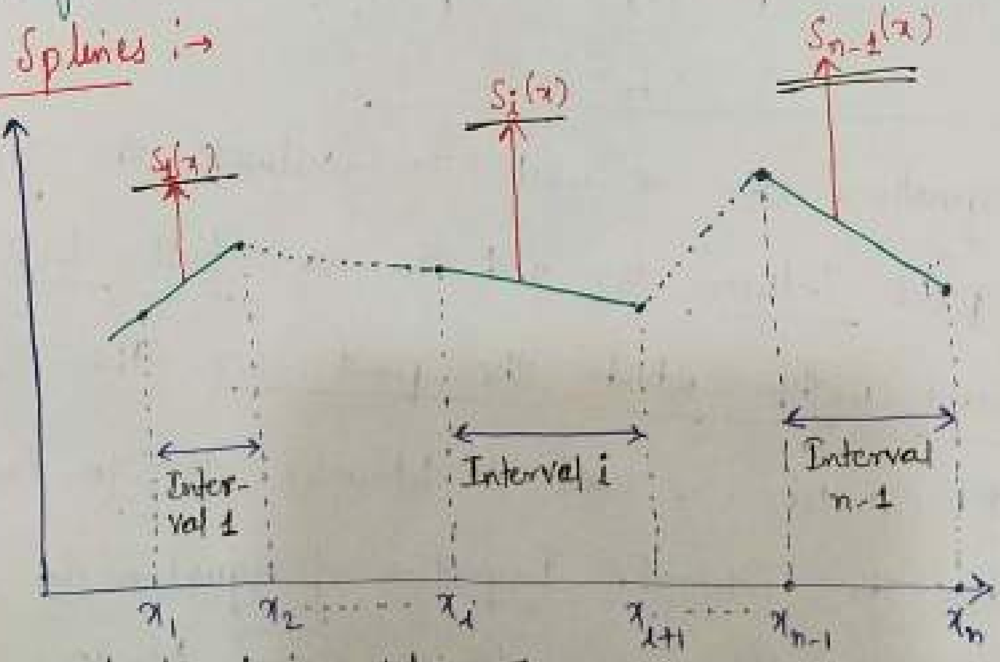
Splines And Piecewise Interpolation

In Lagrange and Newton's divided difference interpolation, $(n-1)$th - order polynomials were used to interpolate between n- data points. There are cases where polynomials can lead to erroneous results because of round - off errors and oscillations. An alternative approach is to apply lower - order polynomials in a piecewise fashion to subsets of data points. Such connecting polynomials are called spline functions. For example, third order curves employed to connect each pair of data points are called cubic splines.

→ The spline usually provides a superior approximation of the behaviour of functions that have local, abrupt changes.

Linear Splines :→



Notations used to derive splines. There are $(n-1)$ intervals and n data points.

For n data points ($i = 1, 2, \ldots, n$), there are $(n-1)$ intervals. Each interval i has its own spline function $S_i(x)$. For linear splines, each function is merely the straight line connecting the two points at each end of the interval, which is formulated as

$$S_i(x) = a_i + b_i(x - x_i) \quad - (1)$$

where a_i is the intercept which is defined as

$$a_i = f_i \quad - (2)$$

and b_i is the slope of the straight line connecting the points:

$$b_i = \frac{f_{i+1} - f_i}{x_{i+1} - x_i} \quad - (3)$$

where f_i is shorthand for $f(x_i)$. Putting equations (2) and (3) into equation (1), we get:

$$S_i(x) = f_i + \frac{f_{i+1} - f_i}{x_{i+1} - x_i}(x - x_i). \quad - (4)$$

These equations can be used to evaluate the function at any point between x_1 and x_n by first locating the interval within which the point lies. Then the approximate equation is used to determine the function value within the interval. Inspection of equation (4) indicates that the linear spline amounts to using Newton's first-order polynomial to interpolate within each interval. ②

Example. Fit the data in Table with first order splines; that is, linear splines. Evaluate the function at $x=5$.

i	x_i	f_i
1	3.0	2.5
2	4.5	1.0
3	7.0	2.5
4	9.0	0.5

Solution. Since there are 4 data points, so $(4-1)=3$ intervals $\Rightarrow$ Hence, correspondingly 3 linear spline functions; that is $s_1(x)$, $s_2(x)$ and $s_3(x)$.

using equation ④, we can write

$$s_1(x) = f_1 + \frac{f_2 - f_1}{x_2 - x_1}(x - x_1)$$

$$= 2.5 + \frac{1 - 2.5}{4.5 - 3.0}(x-3)$$

$$= 2.5 - 1(x-3) = 2.5 - x + 3$$

$$\Rightarrow \quad s_1(x) = -x + 5.5$$

Also $$s_2(x) = f_2 + \frac{f_3 - f_2}{x_3 - x_2}(x - x_2)$$

$$= 1 + \frac{2.5 - 1}{7 - 4.5}(x - 4.5)$$

$$= 1 + (0.6)(x - 4.5) = 1 + 0.6x - 2.7$$

$\Rightarrow \quad S_2(x) = 0.6x - 1.7$

And $\quad S_3(x) = f_3 + \dfrac{f_4 - f_3}{x_4 - x_3}(x - x_3)$

$$= 2.5 + \frac{0.5 - 2.5}{9 - 7}(x - 7)$$

$$= 2.5 - 1(x - 7) = 2.5 - x + 7$$

$\Rightarrow \quad S_3(x) = -x + 9.5$

Now to find value at $x = 5$, first we have to find the interval in which $x = 5$ lies.

$\Rightarrow$ $x = 5$ lies in second interval $[4.5, 7]$

$\Rightarrow$ $S_2(x)$ will be used to find the value at $x = 5$.

Hence, the value at $x = 5$ is

$S_2(x) = 0.6x - 1.7$

putting $x = 5$

$S_2(5) = 0.6 \times 5 - 1.7 = 1.3$.

The primary disadvantage of first-order splines is that they are not smooth. In essence, at the data points where two splines meet (called a knot), the slope changes abruptly. In formal terms, the first derivative of the function is discontinuous at these points. This deficiency is overcome by using higher-order polynomial splines that ensures smoothness at the knots by equating derivatives at ~~a~~ these points. ③

An alternate method to derive linear spline equation in interval $[x_i, x_{i+1}]$ is to write the equation of line connecting two points (x_i, y_i) and (x_{i+1}, y_{i+1}) which

$$y - y_i = m(x - x_i) \quad \text{(m is the slope of line)}$$

$$\Rightarrow y - y_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}(x - x_i)$$

$$\Rightarrow y = y_i + \frac{y_{i+1} - y_i}{x_{i+1} - x_i}(x - x_i)$$

putting $y_i = f_i$, ~~and~~ $y_{i+1} = f_{i+1}$ and writing y as $s_i(x)$, we get equation ④; that is

$$\Rightarrow s_i(x) = f_i + \frac{f_{i+1} - f_i}{x_{i+1} - x_i}(x - x_i).$$

Cubic Splines :→

Cubic splines are most frequently used in practice. Quadratic or higher-order splines are not used because they tend to exhibit the instabilities inherent in higher-order polynomials. Cubic splines are preferred because they provide the simplest representation that exhibit the desired appearance of smoothness.

The objective in cubic splines is to derive a third-order polynomial for $\boxed{\text{each}}$ interval between knots, as represented generally by

$$S_i(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3$$

Thus, for n-data points ($i = 1, 2, \ldots, n$), there are $(n-1)$ intervals and $4(n-1)$ unknown coefficients (a_i, b_i, c_i and d_i → 4, for each interval) to evaluate. Consequently, $4(n-1)$ conditions are required for their evaluation.

The polynomials are set up so that the functions pass through the points and the first derivatives at the knot are equal. In addition to these, conditions are developed to ensure that the second derivatives at the knot are also equal. This greatly enhances the ~~fits~~ fit's smoothness.

After these conditions are developed, two ~~add~~ additional (4) conditions are required to obtain the solution. In that case, we had to arbitrarily specify a zero second derivative for the first interval. The ~~last condition~~

We can also put a zero second derivative for the last interval as well as the last condition.

The visual interpretation of these conditions is that the function becomes a straight line at the end nodes. Specification of such an end condition leads to what is termed as "natural" spline.

There are variety of other end conditions that can be specified. Two of the more popular are clamped condition and the not-a-knot conditions.

Derivation of Cubic Splines :→

The first condition is that the spline must pass through all the data points.

$$\Rightarrow S_i(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3 \quad - (5)$$

this must pass through point (x_i, f_i)

$$\Rightarrow f_i = a_i + b_i(x_i-x_i) + c_i(x_i-x_i)^2 + d_i(x_i-x_i)^3$$

$$\Rightarrow \boxed{f_i = a_i} \quad - (6)$$

Therefore, the constant in each cubic spline must be equal to the value of the dependent variable at the beginning of the interval. Substituting eq. 6 into equation 5, we get

$$S_i(x) = f_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3 \quad - 7$$

Next, we will apply the condition that each of the cubics must join at the knots; that is $S_i(x)$ must also pass through the point (x_{i+1}, f_{i+1}).. For knot $i+1$, this can be represented as

$$f_{i+1} = f_i + b_i(x_{i+1}-x_i) + c_i(x_{i+1}-x_i)^2 + d_i(x_{i+1}-x_i)^3$$

$$\Rightarrow f_i + b_i h_i + c_i h_i^2 + d_i h_i^3 = f_{i+1} \quad - 8$$

where $h_i = x_{i+1} - x_i$

The first derivatives at the interior nodes must be equal. Equation 7 is differentiated to yield

$$S_i'(x) = b_i + 2c_i(x-x_i) + 3d_i(x-x_i)^2 \quad - 9$$

The equivalence of the derivatives at the interior node, ~~at~~ $i+1$ can therefore be written as

$$b_i + 2c_ih_i + 3d_ih_i^2 = b_{i+1} \quad - 10$$

by putting (x_{i+1}, f_{i+1}) in $S_i'(x)$ and $S_{i+1}'(x)$ and then equating both $S_i'(x)$ and $S_{i+1}'(x)$.

The second derivative at the interior nodes must also be equal. Eq. 9 can be differentiated to yield

$$S_i''(x) = 2c_i + 6d_i(x-x_i) \quad - 11$$

The equivalence of the second derivative at an interior node, i+1 can therefore be written as

$$c_i + 3d_i h_i = c_{i+1} \quad - (12)$$

by putting (x_{i+1}, f_{i+1}) in $S_i''(x)$ and $S_{i+1}''(x)$ and then equating the second derivatives: that is

$$\cancel{S_i''(x_{i+1}) = 2c_i}$$

$$S_i''(x_{i+1}) = 2c_i + 6d_i h_i \quad - (a)$$

$$S_{i+1}''(x_{i+1}) = 2c_{i+1} + 3d_{i+1}(x_{i+1} - x_{i+1}) = 2c_{i+1} \quad - (b)$$

Equating (a) and (b), we get equation (12).

Next, we can solve eq (12) for d_i:

$$\boxed{d_i = \frac{c_{i+1} - c_i}{3h_i}} \quad - (13)$$

This can be substituted into eq (8) to give

$$f_i + b_i h_i + \frac{h_i^2}{3}(2c_i + c_{i+1}) = f_{i+1} \quad - (14)$$

Equation (13) can also be substituted into eq (10) to give

$$b_{i+1} = b_i + h_i(c_i + c_{i+1}) \quad - (15)$$

Equation (14) can also be solved for:

$$\boxed{b_i = \frac{f_{i+1} - f_i}{h_i} - \frac{h_i}{3}(2c_i + c_{i+1})} \quad - (16)$$

The index of this equation can be reduced by 1:

$$b_{i-1} = \frac{f_i - f_{i-1}}{h_{i-1}} - \frac{h_{i-1}}{3}(2c_{i-1} + c_i) \quad - (17)$$

The index of equation 15 can also be reduced by 1:

$$b_i = b_{i-1} + h_{i-1}(c_{i-1} + c_i) \quad - 18$$

Equations 16 and 17 can be substituted into eq 18 and the result simplified to yield

$$\frac{f_{i+1} - f_i}{h_i} - \frac{h_i}{3}(2c_i + c_{i+1}) = \frac{f_i - f_{i-1}}{h_{i-1}} - \frac{h_{i-1}}{3}(2c_{i-1} + c_i) + h_{i-1}(c_{i-1} + c_i)$$

$$\Rightarrow 3\left(\frac{f_{i+1} - f_i}{h_i}\right) - 3\left(\frac{f_i - f_{i-1}}{h_{i-1}}\right) = h_i(2c_i + c_{i+1}) - h_{i-1}(2c_{i-1} + c_i) + 3h_{i-1}(c_{i-1} + c_i)$$

$$\Rightarrow h_{i-1}c_{i-1} + 2c_i(h_{i-1} + h_i) + h_i c_{i+1} = 3\left(\frac{f_{i+1} - f_i}{h_i}\right) - 3\left(\frac{f_i - f_{i-1}}{h_{i-1}}\right) \quad - 19$$

This equation can be made a little more concise by recognising that the terms on the right side are finite differences; that is

$$f[x_i, x_{i+1}] = \frac{f_{i+1} - f_i}{x_{i+1} - x_i} = \frac{f_{i+1} - f_i}{h_i}$$

Therefore, equation 19 can be written as

$$h_{i-1}c_{i-1} + 2c_i(h_{i-1} + h_i) + h_i c_{i+1} = 3\left(f[x_{i+1}, x_i] - f[x_i, x_{i-1}]\right) \quad - 20$$

Equation 20 can be written for the interior knots $i = 2, 3, \ldots, n-2$, which results in $n-3$ simultaneous tridiagonal equations with $(n-1)$ unknown coefficients

$c_1, c_2, \ldots, c_{n-1}$. Therefore, if we have two additional conditions, we can solve for the c's. Once this is done, the equations (16) and (13) can be used to determine the remaining coefficients, b and d.

The two additional end conditions can be formulated in a number of ways. One common approach, the natural spline, assumes that the second derivative at the end knots are equal to zero. The second derivative at the first node (eq (11)) can be set to zero as in

$$S_i''(x_i) = 2c_i + 6d_i(x - x_i) \quad \text{putting } i = 1$$

$$\Rightarrow S_1''(x) = 2c_1 + 6d_1(x - x_1)$$

$$\Rightarrow S_1''(x_1) = 2c_1 + 6d_1(x_1 - x_1) = 0$$

$$\Rightarrow \boxed{c_1 = 0}$$

Thus, this condition amounts to setting c_1 equal to zero.

The same evaluation can be made at the last node:

$$S_{n-1}''(x_n) = 0 = 2c_{n-1} + 6d_{n-1}h_{n-1} \quad \text{— (21)}$$

Recalling equation (12), we can conveniently define an extraneous parameter c_n, in which case eq. (21) becomes

$$c_{n-1} + 3d_{n-1}h_{n-1} = \boxed{c_n = 0}$$

Thus, to impose a zero second derivative at the last node, we let $c_n = 0$

The final equations can now be written in matrix form as

$$\begin{bmatrix} 1 & & & & & \\ h_1 & 2(h_1+h_2) & h_2 & & & \\ & h_2 & 2(h_2+h_3) & h_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_{n-1} \\ C_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 3(f[x_3,x_2] - f[x_2,x_1]) \\ 3(f[x_4,x_3] - f[x_3,x_2]) \\ \vdots \\ 3(f[x_n,x_{n-1}] - f[x_{n-1},x_{n-2}]) \\ 0 \end{bmatrix} \quad - (22)$$

The system is tridiagonal and hence effective to solve.

Example fit the data given in table with cubic splines. Evaluate the function at $x = 5$.

i	x_i	f_i
1	3	2.5
2	4.5	1
3	7	2.5
4	9	0.5

The first step is to employ equation (22) to generate (7) set of simultaneous equations to determine the c coefficients

$$\begin{bmatrix} 1 & & & \\ h_1 & 2(h_1+h_2) & h_2 & \\ & h_2 & 2(h_2+h_3) & h_3 \\ & & & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3(f[x_3,x_2]-f[x_2,x_1]) \\ 3(f[x_4,x_3]-f[x_3,x_2]) \\ 0 \end{bmatrix}$$

The necessary function and interval width values are

$f_1 = 2.5$ $\qquad h_1 = 4.5 - 3 = 1.5$

$f_2 = 1$ $\qquad h_2 = 7 - 4.5 = 2.5$

$f_3 = 2.5$ $\qquad h_3 = 9 - 7 = 2$

$f_4 = 0.5$

$$\Rightarrow f[x_2,x_1] = \frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{1-2.5}{1.5} = -1$$

$$\Rightarrow f[x_3,x_2] = \frac{f_3-f_2}{h_2} = \frac{2.5-1}{2.5} = 0.6$$

$$\Rightarrow f[x_4,x_3] = \frac{f_4-f_3}{h_3} = \frac{0.5-2.5}{2} = -1$$

These can be substituted to yield

$$\begin{bmatrix} 1 & & & \\ 1.5 & 8 & 2.5 & \\ & 2.5 & 9 & 2 \\ & & & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4.8 \\ -4.8 \\ 0 \end{bmatrix}$$

These equations yield:

$c_1 = 0$ $\qquad c_2 = 0.839543726$

$c_3 = -0.766539924$ $\qquad c_4 = 0$.

Now using equations (16) and (13); that is

$$d_i = \frac{c_{i+1}-c_i}{3h_i} \quad \text{for } i = 1, 2, 3. \quad \text{and}$$

$$b_i = \frac{f_{i+1} - f_i}{h_i} - \frac{h_i}{3}(2c_i + c_{i+1}) \quad \text{for } i = 1, 2, 3$$

to compute b's and d's we get

$b_1 = -1.41977$ $\qquad d_1 = 0.18656$

$b_2 = -0.16046$ $\qquad d_2 = -0.21414$

$b_3 = 0.02205$ $\qquad d_3 = 0.12776$

These result, along with the value of a's (that is $a_i = f_i$, for $i = 1, 2, 3$) can be substituted into

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

for $i = 1, 2, 3$, to develop the cubic spline for each interval:

$$S_1(x) = 2.5 - 1.41977(x-3) + 0.18656(x-3)^3$$

$$S_2(x) = 1 - 0.16046(x-4.5) + 0.83954(x-4.5)^2 - 0.21414(x-4.5)^3$$

$$S_3(x) = 2.5 + 0.02205(x-7) - 0.76654(x-7)^2 + 0.12776(x-7)^3$$

The three equations can then be employed to compute values within each interval. For example, the value at $x = 5$, which falls within the second interval, is calculated as

$$S_2(5) = 1 - 0.16046(5-4.5) + 0.83954(5-4.5)^2 - 0.21414(5-4.5)^3$$

$$\Rightarrow S_2(5) = 1.1028875 \approx 1.10289$$

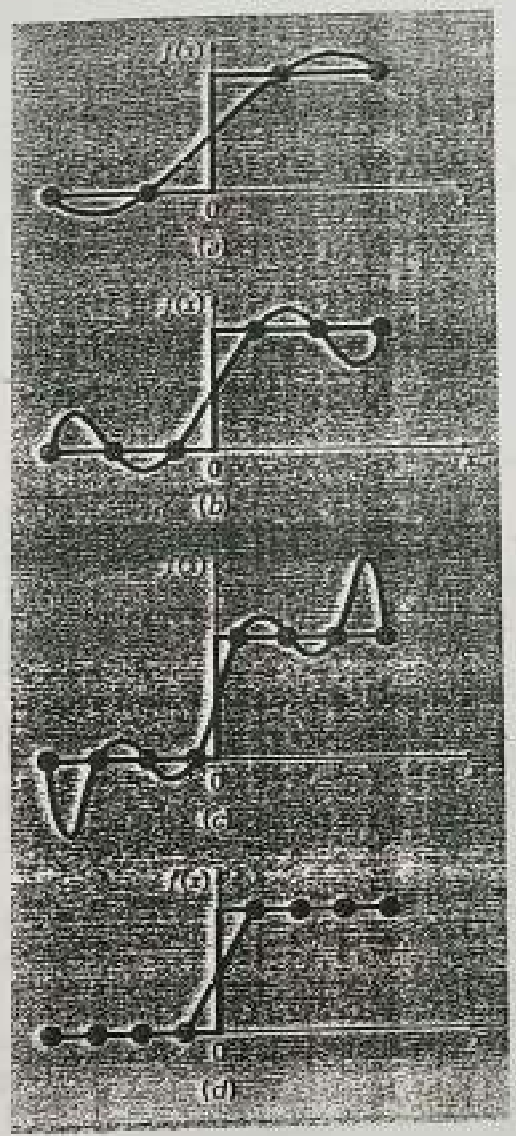


FIGURE 16.1
A visual representation of a situation where splines are superior to higher-order interpolating polynomials. The function to be fit undergoes an abrupt increase at $x = 0$. Parts (a) through (c) indicate that the abrupt change induces oscillations in interpolating polynomials. In contrast, because it is limited to straight-line connections, a linear spline (d) provides a much more acceptable approximation.

adjacent cubic equations are visually smooth. On the surface, it would seem that the third-order approximation of the splines would be inferior to the seventh-order expression. You might wonder why a spline would ever be preferable.

Figure 16.1 illustrates a situation where a spline performs better than a higher-order polynomial. This is the case where a function is generally smooth but undergoes an abrupt change somewhere along the region of interest. The step increase depicted in Fig. 16.1 is an extreme example of such a change and serves to illustrate the point.

Figure 16.1*a* through *c* illustrates how higher-order polynomials tend to swing through wild oscillations in the vicinity of an abrupt change. In contrast, the spline also connects the points, but because it is limited to lower-order changes, the oscillations are kept to a

16.1 Given the data

x	1	2	2.5	3	4	5
$f(x)$	1	5	7	8	2	1

Fit this data with (a) a cubic spline with natural end conditions, (b) a cubic spline with not-a-knot end conditions, and (c) piecewise cubic Hermite interpolation.

16.2 A reactor is thermally stratified as in the following table:

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