$\Rightarrow \forall \alpha \in R$ & $\forall$ $x \in W$

$\Rightarrow \alpha x \in$ $\Rightarrow \alpha x \in W$

now let $x, y \in W \Rightarrow x = [a_1, a_1, a_1, b_1]$

& $y = [a_2, a_2, a_2, b_2]$

$x+y = [a_1, a_1, a_1, b_1] + [a_2, a_2, a_2, b_2]$

$= [a_1, a_1+a_2, a_1 b_1 + b_2] \in R^4$

$\Rightarrow x+y \in$ $W \Rightarrow x+y \in W$

$\Rightarrow$ W is subspace of R^4

Qn Show that the set of vectors of the form $[a, b, \frac{a}{2} - b]$ in R^3 is a subspace of R^3 under usual operations i.e. Addition & scalar multiplication.

Sol: let $W = \{[a, b, \frac{a}{2} - b] \; a, b \in R\}$

To show W is subspace of R^3

let $\forall \alpha \in R$ & $x \in W \Rightarrow$

$x = [a, b, \frac{a}{2} - b]$

$\Rightarrow \alpha x = \alpha[a, b, \frac{a}{2} - b]$

$= [\alpha a, \alpha b, \alpha \frac{a}{2} - \alpha b]$

always in R^3

$\Rightarrow \forall \alpha \in R$ & $\forall x \in W \Rightarrow \alpha x \in W$

ii) $\forall x, y \in W \Rightarrow x = [a_1, b_1, \frac{a_1}{2} - b_1]$

And $y = [a_2, b_2, \frac{a_2}{2} - b_2]$

$x - y = [a_1, b_1, \frac{a_1}{2} - b_1] - [a_2, b_2, \frac{a_2}{2} - b_2]$

$= [a_1 - a_2, b_1 - b_2, (\frac{a_1 - a_2}{2}) - (b_1 - b_2)]$

$\in W$

$\Rightarrow$ If $\forall x, y \in W \Rightarrow x - y \in W$

$\Rightarrow$ W is subspace of R^3

Qn Let V be the vector space of all real-valued functions on R with the usual operations of function addition and scalar multiplication. Show that the set

$W = \{f : R \to R \;\; f(\frac{1}{2}) = 0\}$

is subspace of V.

Sol: let $\alpha \in R$ & $f \in W$

$\Rightarrow f(1/2) = 0$

always $\alpha f(v_2) = \alpha 0 = 0$

$\Rightarrow \alpha f(v_2) = 0$

Now let $f, g \in W \Rightarrow f(v_2) = 0$
and $g(v_2) = 0$

$f+g = f(v_2) + g(v_2) = 0+0 = 0$

$\Rightarrow f+g \in W$

$\Rightarrow \forall f, g \in W \Rightarrow f+g \in W$

$\Rightarrow W$ is subspace of V

<u>Thm</u> Let V be a vector space and W_1 & W_2 be subspace of V then $W_1 \cap W_2$ is also subspace of V

<u>Proof</u> $\because W_1$ & W_2 are subspace of $V \Rightarrow W_1 \neq \phi$ & $W_2 \neq \phi$

and $0 \in W_1$ & $0 \in W_2$

$\Rightarrow 0 \in W_1 \cap W_2$

$\Rightarrow W_1 \cap W_2 \neq \phi$

Now to show $W_1 \cap W_2$ is closed under vector addition

Let x, y be any vectors in $W_1 \cap W_2$

$\Rightarrow x, y \in W_1$ and $x, y \in W_2$

$\Rightarrow x+y \in W_1$ and $x+y \in W_2$

$\Rightarrow x+y \in W_1 \cap W_2$

$\Rightarrow \forall x, y \in W_1 \cap W_2 \Rightarrow x+y \in W_1 \cap W_2$

Next to show $W_1 \cap W_2$ closed scalar multiplication

Let $x \in W_1 \cap W_2$ and $\alpha \in R$
to show $\alpha x \in W_1 \cap W_2$

Since $x \in W_1 \cap W_2 \Rightarrow x \in W_1$
and $x \in W_2$

$\Rightarrow \alpha x \in W_1$ and $\alpha x \in W_2$
[$\because W_1$ and W_2 subspaces]

$\Rightarrow \alpha x \in W_1 \cap W_2$

$\Rightarrow \forall x, y \in W_1 \cap W_2 \Rightarrow x+y \in W_1 \cap W_2$

$\Rightarrow W_1 \cap W_2$ is vector space of V

$\Rightarrow$ Intersection of two subspace is also subspace

union of two subspaces need not be subspace

Counter example [Do your self]

Qn $W = \{(x,y) : x \geq 0\}$ is subspace of $V = \mathbb{R}^2$

Soln W is not subspace

let $w_1 \in W \Rightarrow w_1 = (x,y)$
$= (1,1)$ $\because x \geq 0$
fixed $x = 1$
$y = 1$

let $\alpha \in \mathbb{R}$ $\alpha = -1$ (fixed)

$\alpha w_1 = -1(1,1) = (-1,-1)$
$\hookrightarrow -1 \not\geq 0$

$\Rightarrow \alpha w_1 \notin W$

$\Rightarrow$ W is not subspace

Qn $W = \{(x,y,z) : x \geq y \geq z\}$
Is W subspace?
[Do your self]

Qn $W = \{(a,b,1) : a,b \in \mathbb{R}\}$
is subspace ? over $\mathbb{R}^3$

Soln $\because$ zero vector in W
is $(0,0,0)$ not ~~but~~ in W

$\because (a,b,1) \in W$
vector in W is type of
$(a,b,1)$
$\longrightarrow 1$ never 0

$\Rightarrow (0,0,1)$ get in W but
$(0,0,0)$ never in W

$\Rightarrow$ W is not subspace $\mathbb{R}^3$

Qn $W = \{(x,y) : x^2+y^2 = 0\}$
show that W is subspace
of $\mathbb{R}^2$ [Do your self]

Qn $W = \{(x,y,z) : x-y = 1\}$
$\Rightarrow$ Is W is subspace of $\mathbb{R}^3$?

Qn $W = \{p(x) : p(x)$ is a poly. of
two degree with $p(1) \geq 0\}$

Is W is subspace over $\mathbb{R}$?

Qn If x and y are in R^n then prove that
$$\|3x+5y\| \le 5(\|x\|+\|y\|)$$

Qn Calculate the total work performed by a force $f = [3,2,1]$ on an object which causes a displacement $d = [-1,6,-3]$.

Qn Calculate the total work performed if a force of 26 Newton is exerted on an object in the direction of the vector $[-3,4,5]$ and that the objects travels 10 meter in the direction of the vector $[-1,2,2]$

| Chapter complete |

chapter - 5 [finite-dimensional vector spaces]

LECTURE 1

Def^n A vector space over $\mathbb{R}$ is a set V with two operations, vector addition and scalar multiplication such that following properties hold.

① $\forall\, u, v \in V \Rightarrow u+v \in V$ [closure under addition]

② $\forall\, \alpha \in \mathbb{R}$ and $\forall\, v \in V \Rightarrow \alpha v \in V$ [closure under scalar multiplication]

③ $\forall\, u, v \in V \Rightarrow u+v = v+u$ [commutative]

④ $\forall\, u, v, w \in V \Rightarrow (u+v)+w = u+(v+w)$ [Associative]

⑤ $\exists$ an elt $0 \in V$ s.t. $\forall\, v \in V$
$v+0 = 0+v$
[existence of additive identity]

⑥ $\forall\, v \in V$ $\exists$ an elt $-v \in V$ s.t.
$v+(-v) = 0 = (-v)+v$
[existence of additive inverse]

⑦ $\forall\, a \in \mathbb{R}$ $\forall\, u, v \in V$
$\Rightarrow a(u+v) = au + av$
[Distributive law over scalar multiplication]

(8) $\forall\, a, b \in R$ and $\forall\, v \in V \Rightarrow (a+b)v = av + bv$
[Distributive law over ~~scalar~~ addition]

(9) $\forall\, a, b \in R$ and $v \in V \Rightarrow (ab)v = a(bv)$
[Associativity of scalar multiple]

(10) $\forall\, v \in V\ \exists\ 1 \in V$ s.t. $1 \cdot v = v = v \cdot 1$

Qn 1 R^n is the set of n-tuples $(x_1, x_2, \dots x_n)$ of real numbers. Then to show R^n is vector space over R

Sol: Let $u = (x_1, x_2, \dots x_n) \in R^n$
$v = (y_1, y_2, \dots y_n) \in R^n$

$u+v = (x_1, x_2, \dots x_n) + (y_1, y_2, \dots y_n)$

$= (x_1+y_1, x_2+y_2, \dots x_n+y_n) \in R^n$

$\Rightarrow u+v \in R^n$

(i) Let $\alpha \in R$ & $v = (y_1, y_2, \dots y_n) \in R^n$

$\alpha v = \alpha(y_1, y_2, \dots y_n) = (\alpha y_1, \alpha y_2, \dots \alpha y_n) \in R^n$

$\Rightarrow \alpha v \in R^n$

(ii) Let $u = (x_1, x_2, \dots x_n) \in R^n$ and
$v = (y_1, y_2, \dots y_n) \in R^n$

$u+v = (x_1, x_2, \dots x_n) + (y_1, y_2, \dots y_n)$
$= (x_1+y_1, x_2+y_2, \dots x_n+y_n)$

$= (y_1+x_1, y_2+x_2, \dots y_n+x_n)$
[$\because a+b = b+a \quad \forall\, a, b \in R$]

$= (y_1, y_2, \dots y_n) + (x_1, x_2, \dots x_n)$

$= v+u$

$\Rightarrow u+v = v+u \quad \forall\, u, v \in V = R^n$

(iii) Let $u = (x_1, x_2, \dots x_n)$
$v = (y_1, y_2, \dots y_n)$
$w = (z_1, z_2, \dots z_n)$
$u, v, w \in R^n$

$(u+v)+w = [(x_1, x_2, \dots x_n) + (y_1, \dots y_n)] + (z_1, \dots z_n)$

$= (x_1+y_1, x_2+y_2, \dots x_n+y_n) + (z_1, \dots z_n)$

$= (x_1 + y_1 + z_1, x_2 + y_2 + z_2, \ldots x_n + y_n + z_n)$

$= (x_1, x_2 \ldots x_n) + (y_1 + z_1 \ldots y_n + z_n)$

$= (x_1, x_2 \ldots x_n) + [(y_1 + z_1 \ldots y_n + z_n)]$

$= (x_1, x_2 \ldots x_n) + [(y_1, y_2 \ldots y_n) + (z_1, z_2 \ldots z_n)]$

$= V + (U + W)$

$\Rightarrow (U + V) + W = U + (V + W) \;\; \forall U, V, W \in R^n$

(2) We know that zero vector always in R^n
$\Rightarrow (0, 0 \ldots 0) \in R^n$

Let $V = (x_1, x_2 \ldots x_n) \in R^n$ then $\exists$ $(0, 0, \ldots 0) \in R^n$

Let $(0, 0 \ldots 0) + (x_1 \ldots x_n) = \cancel{(x_1 \ldots x_n)}$

$= (x_1 + 0, x_2 + 0 \ldots x_n + 0)$

$= (0 + x_1 \ldots 0 + x_n)$
$= (0, 0 \ldots 0) + (x_1, x_2 \ldots x_n)$

$\Rightarrow \exists$ addition identity ~~of impo~~

(3) Existence of Inverse (addition)

Let $(x_1, x_2 \ldots x_n) \in R^n$ then there always exist $(-x_1, -x_2 \ldots -x_n) \in R^n$

Let $(x_1, x_2 \ldots x_n) + (-x_1, -x_2 \ldots -x_n)$

$= (x_1 + (-x_1), x_2 + (-x_2) \ldots x_n + (-x_n))$

$= (x_1 - x_1, x_2 - x_2 \ldots x_n - x_n)$

$= (0, 0, \ldots 0) = 0$ (zero vector)

Similarly $(-x_1, -x_2 \ldots -x_n) + (x_1 \ldots x_n)$
$=$ zero vector

(4) Let $\alpha \in R$ & $U, V \in R^n$ where

$U = (x_1 \ldots x_n)$ $\quad V = (y_1 \ldots y_n)$

$\alpha(U + V) = \alpha[(x_1 \ldots x_n) + (y_1 \ldots y_n)]$

$= \alpha[(x_1 + y_1 \ldots x_n + y_n)]$

$= (\alpha(x_1 + y_1) \ldots \alpha(x_n + y_n))$
$= [\alpha x_1 + \alpha y_1 \ldots \alpha x_n + \alpha y_n]$

$= (\alpha x_1, \alpha x_2 \ldots + \alpha x_n) + (\alpha y_1 \ldots \alpha y_n)$

$= \alpha u + \alpha v$

(8) let $\alpha, \beta \in R$ & $v \in R^n$
$v = (y_1 \ldots y_n)$

~~To show~~ $(\alpha + \beta) v = \alpha v + \beta v$
(which is trivaly)

~~$\alpha + \beta$~~

(9) let $\alpha, \beta \in R$ & $v \in R^n$
$v = (y_1 \ldots y_n)$

$(\alpha\beta) v = \alpha(\beta v)$ (which is trivaly)

(10) let $v \in V$ where $v = (y_1 \ldots y_n)$
then always exist $1 \in R$ s.t
$1 v = v$

$\Rightarrow R^n$ is vector space over R

Q: P_n is the set of all polynomials with real coefficients and degree $\leq n$. Prove that P_n is vector space over Real numbers

~~Sol^n let $p(x) \in P_n(R)$ & $q(y) \in P_n(R)$~~
~~$\Rightarrow p(x) = a_0 + a_1 x \ldots + a_n x^n \quad a_i \in R$~~
~~$q(y) = b_0 + b_1 y \ldots b_n y^n \quad b_i \in R$~~
~~$p(x) + q(y) = (a_0 + a_1 x \ldots + a_n x^n) + (b_0 + b_1 y \ldots b_n y^n)$~~
~~$= (a_0 + b_0) + ($~~

Sol^n let $p(x)$ & $q(x) \in P_n$ over R

$\Rightarrow p(x) = a_0 + a_1 x \ldots + a_n x^n$
$q(x) = b_0 + b_1 x \ldots + b_n x^n$

$p(x) + q(x) = (a_0 + a_1 x \ldots + a_n x^n) + (b_0 + b_1 x \ldots + b_n x^n)$

$= (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) x^2 \ldots + (a_n + b_n) x^n \in P_n$ over R

$\Rightarrow p(x) + q(x) \in P_n(R)$

② $p(x), q(x) \in P_n(R)$

$p(x)+q(x) = (a_0+a_1x \dots + a_nx^n) + (b_0+b_1x \dots + b_nx^n)$

$= (a_0+b_0)+(a_1+b_1)x \dots + (a_n+b_n)x^n$

$= (b_0+a_0)+(b_1+a_1)x \dots + (b_n+a_n)x^n$

$[\because a_i, b_j \in R \Rightarrow a_i+b_j = b_j+a_i,\ 0\le i\le n,\ 0\le j\le n]$

$= [b_0+b_1x+b_2x^2 \dots + b_nx^n] + [a_0+a_1x \dots + a_nx^n]$

$= q(x)+p(x)$

$\Rightarrow p(x)+q(x) = q(x)+p(x)$

③ Let $\alpha \in R$ & $p(x) \in P_n(R)$

$p(x) = a_0+a_1x \dots + a_nx^n$

$\alpha p(x) = \alpha(a_0+a_1x \dots + a_nx^n)$

$= \alpha a_0 + (\alpha a_1)x \dots + (\alpha a_n)x^n \in P_n(R)$ $[\because \alpha a_i \in R,\ 0\le i\le n]$

$\Rightarrow \alpha p(x) \in P_n(R)$

④ Let $p(x), q(x)$ and $r(x) \in P_n(R)$

$\Rightarrow p(x) = a_0+a_1x \dots + a_nx^n$ $a_i \in R$ $0\le i\le n$

$q(x) = b_0+b_1x \dots + b_nx^n$ $b_i \in R$ $0\le i\le n$

$r(x) = c_0+c_1x \dots + c_nx^n$ $c_i \in R$ $0\le i\le n$

$(p(x)+q(x))+r(x)$

$= [(a_0+a_1x \dots + a_nx^n)+(b_0 \dots + b_nx^n)] + (c_0+c_1x \dots + c_nx^n)$

$= ((a_0+b_0)+(a_1+b_1)x \dots + (a_n+b_n)x^n) + (c_0+c_1x \dots + c_nx^n)$

$= (a_0+b_0+c_0)+(a_1+b_1+c_1)x + (a_2+b_2+c_2)x^2 + (a_n+b_n+c_n)x^n$

$= [a_0+a_1x+a_2x \dots + a_nx^n] + (b_0+c_0) + (b_1+c_1)x \dots + (b_n+c_n)x^n$

$= (a_0+a_1x \dots + a_nx^n) + [(b_0+b_1x \dots + b_nx^n) + (c_0 \dots + c_nx^n)]$

$= p(x)+(q(x)+r(x))$

$\Rightarrow \forall\ p(x), q(x), r(x) \in P_n(R)$ then $(p(x)+q(x))+r(x) = p(x)+(q(x)+r(x))$

Notes

(5) $p(x) \in P_n(R)$ & there always zero poly s.t $p(x)+0 = 0+p(x) = p(x)$

(6) $p(x) \in P_n(R)$ & there always exist $-p(x) \in P_n(R)$ s.t

$$p(x)+(-p(x)) = (-p(x))+p(x) = 0$$

(7) let $\alpha, \beta \in R$ (scalar) & $p(x) \in P_n$ then always hold

$$(\alpha+\beta)p(x) = \alpha p(x)+\beta p(x)$$

(8) let $\alpha \in R$ & $p(x), q(x) \in P_n(R)$ then always hold

$$\alpha(p(x)+q(x)) = \alpha p(x)+\alpha q(x)$$

(9) $\forall \alpha, \beta \in R$ & $p(x) \in P_n(R)$

$(\alpha\beta)p(x) = \alpha(\beta p(x))$ always true

(10) $\forall p(x) \in P_n(R)$ then $\exists$ ~~also~~ poly $q(x)=1$ s.t

$$1 \cdot p(x) = p(x) \cdot 1 = p(x)$$

$\Rightarrow$ $P_n(R)$ is vector space

chapter - 5 Vector space

Lecture - 2

(Q) Let ϕ be the set of all real-valued functions, f defined on the interval [0,1] such that

$$f(1/2) = 1$$

check ϕ is vector space over [0,1]?

Sol^n For proving ~~any~~ vector space we shall prove 10 condition. If any one of the condition is fail then not vector space

Here $f(1/2) = 1$ let $f, g \in \phi$

$\Rightarrow f(1/2) = 1$ & $g(1/2) = 1$

$\Rightarrow$ Closure prop if $u, v \in V \Rightarrow u+v \in V$

Now $f(1/2) + g(1/2) = 1+1 = 2 \notin \phi$

$\Rightarrow \phi$ is not vector space over [0,1]

or other way

Existence of Additive Inverse

$f \in \phi \Rightarrow f(1/2) = 1$ we find g s.t

$f + g = 0$

i.e $f(v_2) + g(v_1) = 0$

i.e $f(v_1) = -g(v_2)$ i.e $1 = -g(v_2)$

i.e $g(v_2) = -1 \notin \mathbb{Z}$

⇒ Additive inverse not exist
⇒ F is not vector space

Qn $S = \{A \in M_{22} : |A| = 0\}$ S is a vector space over $\mathbb{R}$ or not

Sol $S = \{A \in M_{22} : |A| = 0\}$

Let $A, B \in M_{22}$ s.t $|A| = 0$, $|B| = 0$

$A = \begin{bmatrix}1 & 2\\ 1 & 2\end{bmatrix}$ $|B| = \begin{bmatrix}3 & 2\\ 9 & 6\end{bmatrix}$

$|A| = 0$ & $|B| = 0$

closure prop if $\forall u, v \in V$

$\Rightarrow u + v \in V$

$A \oplus B = \begin{bmatrix}1 & 2\\ 1 & 2\end{bmatrix} + \begin{bmatrix}3 & 2\\ 9 & 6\end{bmatrix} = \begin{bmatrix}4 & 4\\ 10 & 8\end{bmatrix} = C$ (say)

$|C| = 8\times4 - 10\times4 = -8 \neq 0$

$\Rightarrow A \oplus B \notin S$

⇒ S is not vector space over $\mathbb{R}$ or not

Qn Let $S = \{A \in M_{22} : |A| \neq 0\}$ under the usual operation. S is vector space or not?

Sol closure prop $\forall u, v \in V \Rightarrow u + v \in V$

if $u + v \notin V \Rightarrow V$ is not vector space

Let $A, B \in S \Rightarrow |A| \neq 0$ & $|B| \neq 0$

Let us consider $A = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$ $|A| = 2 \neq 0$

$B = \begin{bmatrix}-1 & 0\\ 0 & 2\end{bmatrix}$ $|B| = -2 \neq 0$

$A \oplus B = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix} + \begin{bmatrix}-1 & 0\\ 0 & 2\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 4\end{bmatrix}$

$|A + B| = \begin{vmatrix}0 & 0\\ 0 & 4\end{vmatrix} = 0 \neq 0$

$\Rightarrow S$ is not vector space over $\mathbb{R}$

Qn The set of all poly of degree 5 is a vector space under the usual operation of addition and scalar multiplication.

Sol^n Let $V = \{$ poly of degree 5 $\}$ over $\mathbb{R}$

Let $p, q \in V \Rightarrow p, q$ are poly of degree 5

$\forall u, v \in V \Rightarrow v+u \in V$

if $v+u \notin V \Rightarrow V$ is not vector space

Let $p(x) = 1 + x^4 - x^5 \in V$
$q(x) = 1 + x^5 \in V$
p & q are the poly of degree 5

$p+q = (1+x^4-\cancel{x^5}) + (1+\cancel{x^5})$

$= 2+x^4$ not a poly of degree 5

$\Rightarrow p+q \notin V \Rightarrow V$ is not vector Space over $\mathbb{R}$

Qn $V = \{ f \in P_3 : f(x_0) = 0 \}$ forms a vector space with standard operation

Sol^n $V = \{ f \in P_3 : f(x_0) = 0 \} = \{ f \in P_3 : f(x_0) = 0 \}$

① Closure prop addition Let $f, g \in V$
$\Rightarrow f(x_0) = 0$ & $g(x_0) = 0$

$f(x_0) + g(x_0) = 0 + 0 = 0$
$\Rightarrow f(x_0) + g(x_0) \in V$
$\Rightarrow$ If $f, g \in V \Rightarrow f+g \in V$

② Closure under Scalar multiplication
Let $\alpha \in \mathbb{R}$ & $f \in V \Rightarrow f(x_0) = 0$
α is any Real number
$\alpha f(x_0) = \alpha \cdot 0 = 0 \Rightarrow \alpha f(x_0) = 0$
$\Rightarrow \alpha f(x_0) \in V$
$\Rightarrow \forall \alpha \in \mathbb{R}, \forall f \in V \Rightarrow \alpha f \in V$

③ Commutative Law of addition
Let $f, g \in V \Rightarrow f(x_0) = 0 = g(x_0)$

$f(x_0) + g(x_0) = 0 + 0 = 0 = g(x_0) + f(x_0)$

$\Rightarrow$ Commutative Law hold

④ Associative Law
Let $f, g, h \in V \Rightarrow f(x_0) = 0 = g(x_0) = h(x_0)$

$[f(v_1)+g(v_1)]+h(v_1) = f(v_1)+[g(v_1)+h(v_1)]$

$\Rightarrow$ which is trivially

⑤ Let $f \in V \Rightarrow f(v_1)=0$
always exist 0 poly in V
$\Rightarrow 0+f(v_1) = f(v_1) = f(v_1)+0$

$\Rightarrow \forall f \in V \; \exists$ an elt 0 in V

⑥ $\forall f \in V \Rightarrow f(v_1)=0$ —(i)
$\Rightarrow -f(v_1)=0$ —(ii)

(i)+(ii) $f(-v_1)-f(v_1)=0$

$\Rightarrow -f(v_1)$ is additive inverse of $f(v_1)$

$\Rightarrow \forall f \in V$ then $\exists -f \in V$ s.t
~~f+(-f~~ $f(v_1)+(-f(v_1))=0$
i.e additive inverse exist

⑦ Distributive Law
$\forall \alpha \in \mathbb{R}$ & $\forall f, g \in V \Rightarrow f(v_1)=g(v_1)=0$

$\alpha(f(v_1)+g(v_1)) = \alpha(0+0)$
$= \alpha.0 = 0$

Now $\alpha f(v_1)+\alpha g(v_1) = \alpha.0+\alpha.0=0$

$\Rightarrow \alpha f(v_1)+\alpha g(v_1)=0$

$\Rightarrow \alpha[f(v_1)+g(v_1)] = \alpha f(v_1)+\alpha g(v_1)$

$\Rightarrow \forall \alpha \in \mathbb{R}$ & $\forall f, g \in V \Rightarrow$
$\Rightarrow \alpha(f+g) = \alpha f+\alpha g$.

⑧ Distributive law
$\forall \alpha, \beta \in \mathbb{R}$ & $\forall f \in V \Rightarrow f(v_1)=0$
Now
$(\alpha+\beta) f(v_1) = (\alpha+\beta).0 = 0$

$\alpha f(v_1)+\beta f(v_1) = \alpha.0+\beta.0=0$

$\Rightarrow (\alpha+\beta) f(v_1) = \alpha f(v_1)+\beta f(v_1)$

⑨ Associativity of scalar multiplication
$\forall \alpha, \beta \in \mathbb{R}$ $\forall f \in V$

$(\alpha\beta) f(v_1) = (\alpha\beta).0 = 0$

$\alpha(\beta f(v_1)) = \alpha(\beta.0)=0$
$\Rightarrow (\alpha\beta) f(v_1) = \alpha(\beta f(v_1))$

⑩ existence of scalar multiplication Identity
$\exists$ always $1 \in V$ s.t $1.f(v_1)=0$

$\Rightarrow V = \{f \in P_3 : f(v_1)=0\}$ is vector space over $\mathbb{R}$

Notes

Qn ~~$V = \{ f \in P_3(\mathbb{R}) \} = \{$ set of poly~~

Qn $V = \{$Set of poly degree $\leq 3\}$ is vector space or not [Do your self]

Qn $V = \{$ All real numbers over complex numbers $\}$

i.e. $V = R(\mathbb{C})$ is vector space or not

Soln $V = R(\mathbb{C})$

let $v \in R$ & $v = 8$ (fixed)

& let $\alpha \in \mathbb{C}$ (field)

& $\alpha = 3i$

$\alpha v = (3i) 8 = 24i \notin R$

$\Rightarrow V = R(\mathbb{C})$ is not vector space

Qn $V = R(R)$ is vector space or not [Do your self]

Qn $V = \mathbb{C}(R)$ is vector space or not [Do your self] vector space

Lecture - 3

Qn $V = \mathbb{C}(R)$ is vector space.

Soln we check all 10 condition.

(1) Let $x, y \in \mathbb{C}(R) \Rightarrow x+y$ is also complex no
$\Rightarrow x+y \in \mathbb{C}(R)$

$\Rightarrow \forall\, x, y \in \mathbb{C}(R) \Rightarrow x+y \in \mathbb{C}(R)$

(2) Let $\alpha \in R$ & $x \in \mathbb{C}(R)$
always $\alpha x \in \mathbb{C}(R)$

(3) $\forall\, x, y \in \mathbb{C}(R)$ then always hold
$x+y = y+x$

(4) $\forall\, x, y, z \in \mathbb{C}(R)$ then always hold
$(x+y)+z = x+(y+z)$

(5) $\forall\, x \in \mathbb{C}(R)$ then always exist $-x$ in $\mathbb{C}(R)$ s.t $x+(-x) = 0$

(6) $\forall\, x \in \mathbb{C}(R)$ always $0 \in \mathbb{C}(R)$
$\Rightarrow x+0 = 0+x = x$

[Rest prop all yourself]

(59)

If $V = F_1(F_2)$ is vector space if $F_1 \subseteq F_2$ and F_1 & F_2 are fields.

Ex $V(F) = \mathbb{Q}(\mathbb{Q}(i))$ is not vector space
$\because \mathbb{Q}(i) \notin \mathbb{Q}$

Ex $V(F) = \mathbb{Q}(i)(\mathbb{Q})$ is always form vector space
$\because \mathbb{Q} \subseteq \mathbb{Q}(i)$ & $\mathbb{Q}, \mathbb{Q}(i)$ are fields

Ex $X = \mathbb{R}^2$ with operations defined as

$v_1 \oplus v_2 = v_1 + v_2$ and $\alpha \odot v = \alpha(Av)$

where $A = \begin{bmatrix} -4 & 1 \\ 7 & -2 \end{bmatrix}$

$X = \mathbb{R}^2$ is vector space?

Sol^n X is closed under $\oplus$ & $\odot$ but for multiplicative Identity

Let $v = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ (fixed)

(61)

$1 \odot v = 1 \cdot \begin{bmatrix} \\ \end{bmatrix}$ $1(Av) = 1\left(\begin{bmatrix} -4 & 1 \\ 7 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right)$

$= 1\begin{bmatrix} -4 \times 2 + 1 \times 5 \\ 7 \times 2 - 2 \times 5 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$\Rightarrow 1 \odot v \neq v$

$\Rightarrow X$ is not vector space.

Thm In any vector space additive inverse of each vector is unique.

Proof Let V be a vector space and $v \in V$ & let w & w' are both additive inverse of v

$\Rightarrow w + v = 0$ and $w' + v = 0$

Now $w = w + 0 = w + (v + w')$
$= (w + v) + w'$ [Asso prop]
$= 0 + w'$ [$\because w + v = 0$]

$\Rightarrow w = w'$
$\Rightarrow$ Additive inverse is unique.

Thm Let V be a vector space then $\forall\, v \in V$ and $\forall\, a \in \mathbb{R}$ then

① $a \cdot 0 = 0$ ③ $(-1)v = -v$

② $0 \cdot v = 0$ ④

④ if $av = 0$ then $a = 0$ or $v = 0$

proof ① To show $a \cdot 0 = 0$

LHS $a \cdot 0 = a \cdot 0 + 0$ [Additive Identity]

$= a \cdot 0 + 0$

$= a \cdot 0 + [a \cdot 0 + (-a \cdot 0)]$ [Additive Inverse]

$= (a \cdot 0 + a \cdot 0) + (-a \cdot 0)$ [Asso prop]

$= a(0+0) + (-a \cdot 0)$ [distributive prop]

$= a \cdot 0 + (-a \cdot 0)$

$= 0$ [∵ $-a \cdot 0$ is additive inverse]

$\Rightarrow \boxed{a \cdot 0 = 0}$

proof ② To show $0 \cdot v = 0$

LHS $= 0 \cdot v$ proof same in ①

$= 0 \cdot v$

proof ③ To show $(-1)v = -v$

~~LHS =~~

Let us consider $v + (-1)v$

$= 1v + (-1)v$ [∵ multiplicative identity]

$= (1 + (-1))v$ [∵ distributive]

$= 0 \cdot v$ [∵ is additive inverse of 1]

$= 0$ [from proof ②]

$\Rightarrow v + (-1)v = 0$

$\Rightarrow (-1)v = -v$ [H.P]

proof ④ Let $av = 0$ and $a \neq 0$

We shall prove $v = 0$

∵ $a \neq 0 \Rightarrow$ there exist an element $a^{-1} \in \mathbb{R}$ such that $a^{-1}a = 1$

Now $v = 1v$ [multiplicative identity]

$= (a^{-1}a)v$ [∵ $a^{-1}a = 1$]

$= a^{-1}(av)$ [Asso prop of scalar multip]

$= a^{-1}(0)$ [∵ $av = 0$]

$= 0$ [from ① proof]

$\Rightarrow v = 0$ if $a \neq 0$

~~if a = 0 ...~~

we define 'addition' and 'scalar multiplication' on $V = \mathbb{R}^+$ as follows

$v_1 \oplus v_2 = v_1 \cdot v_2 \quad \forall v_1, v_2 \in V$

and $a \odot v = v^a \quad \forall a \in \mathbb{R}$ and $v \in V$

now to show this is vector space

Soln Let $v_1, v_2 \in V = \mathbb{R}^+$

$\Rightarrow v_1 \oplus v_2 = v_1 \cdot v_2 \in \mathbb{R}^+$

(ii) Let α be any scalar & $\forall v_1 \in V$

$\Rightarrow \alpha \odot v = v^\alpha$ = also +ve real number

$\Rightarrow v^\alpha \in \mathbb{R}^+$

$\Rightarrow \alpha \odot v \in \mathbb{R}^+$

(iii) Commutative property Let $v_1, v_2 \in V = \mathbb{R}^+$ then $v_1 \oplus v_2 = v_1 \cdot v_2 = v_2 \cdot v_1 = v_2 \oplus v_1$

$\Rightarrow v_1 \oplus v_2 = v_2 \oplus v_1$

(iv) Associative Property

Let $v_1, v_2, v_3 \in V = \mathbb{R}^+$ then

$(v_1 \oplus v_2) \oplus v_3 = (v_1 v_2) \oplus v_3 \quad [\because v_1 \oplus v_2 = v_1 \cdot v_2]$

$= (v_1 \cdot v_2) \cdot v_3 = v_1 \cdot v_2 \cdot v_3$ is a +ve Real number

$= v_1 \cdot (v_2 \cdot v_3)$

$= v_1 \cdot (v_2 \oplus v_3)$

$= v_1 \oplus (v_2 \oplus v_3)$

$\Rightarrow (v_1 \oplus v_2) \oplus v_3 = v_1 \oplus (v_2 \oplus v_3)$

(v) Existence of additive identity

$1 \in \mathbb{R}^+$ is additive Identity

$1 \oplus v = 1 v = v$

and $v \oplus 1 = v 1 = v$

(vi) Existence of additive inverse

For each $v \in \mathbb{R}^+$ the additive inverse of v is $\frac{1}{v} \in \mathbb{R}^+$ because

$v \oplus \frac{1}{v} = v \cdot \frac{1}{v} = 1$ is the zero vector in $\mathbb{R}^+$

(vii) Distributive Property

$\forall a \in \mathbb{R}$ and $v_1, v_2 \in V = \mathbb{R}^+$

$a \odot (v_1 \oplus v_2) = a \odot (v_1 \cdot v_2)$

$= a \odot (v_1 \cdot v_2)$

$= (v_1 \cdot v_2)^a = v_1^a \cdot v_2^a$

$= (a \odot v_1) \cdot (a \odot v_2)$

$a\odot(u_1\oplus u_2) = (a\odot u_1)\cdot(a\odot u_2)$

$= (a\odot u_1)\oplus(a\odot u_2)$

$\Rightarrow a\odot(u_1\oplus u_2) = (a\odot u_1)\oplus(a\odot u_2)$

(VIII) Distributive Property $\forall a,b\in\mathbb{R}^+$ and $u\in V$

$(a+b)\odot u = u^{a+b} = u^a\cdot u^b$

$= (a\odot u)\cdot(b\odot u)$

$= (a\odot u)\oplus(b\odot u)$

$\Rightarrow (a+b)\odot u = (a\odot u)\oplus(b\odot u)$

(IX) Associative Property of scalar multiplication $\forall a,b\in\mathbb{R}$ and $u\in V=\mathbb{R}^+$

$(ab)\odot u = u^{ab} = u^{ba} = (u^b)^a$

$= (b\odot u)^a$

$= a\odot(b\odot u)$

$\Rightarrow (ab)\odot u = a\odot(b\odot u)$

(X) Identity under scalar multiplication

$\because 1\in\mathbb{R}^+$

$1\odot u = u^1 = u$

$\Rightarrow V=\mathbb{R}^+$ is vector space

Vector space

[Lecture-4]

Q: Let V be the vector space set $\mathbb{R}^2$ with operations of addition and scalar multiplication defined by

$[x,y]\oplus[w,z] = [x+w-2,\ y+z+3]$ $\forall [x,y], [w,z]\in V$

and $\alpha\odot[x,y] = [\alpha x-2\alpha+2,\ \alpha y+3\alpha-3]$ $\forall \alpha\in\mathbb{R}$ and $\forall [x,y]\in V$

Show that V is a vector space over $\mathbb{R}$. Find the zero vector in V and additive inverse of each vector V.

Sol: Let $u_1, u_2\in V=\mathbb{R}^2$ & $\alpha\in\mathbb{R}$ where $u_1=[x,y]$, $u_2=[w,z]$

① Closure under addition

$\forall u_1,u_2\in V$ $u_1\oplus u_2 = [x,y]\oplus[w,z]$

$= [x+w-2,\ y+z+3]$ always in $\mathbb{R}^2$

② Closure under scalar multiplication

$\forall \alpha\in\mathbb{R}$ & $\forall u_1\in\mathbb{R}$

$\alpha\odot u_1 = \alpha\odot[x,y] = [\alpha x-2\alpha+2,\ \alpha y+3\alpha-3]$ always in $\mathbb{R}^2$

③ Commutative Prop: Let $u_1=[x,y]$, $u_2=[w,z]\in\mathbb{R}^2=V$

then $v_1 \oplus v_2 = [x,y] \oplus [w,z]$

$= [x+w-2,\ y+z+3]$

$= [w+x-2,\ z+y+3]$

$= [w,z] \oplus [x,y]$

$= v_2 \oplus v_1$

$\Rightarrow v_1 \oplus v_2 = v_2 \oplus v_1$

(4) Associative property: let $v_1 = [x,y]$, $v_2 = [u,v]$, $v_3 = [w,z] \in V$ then

$(v_1 \oplus v_2) \oplus v_3 = ([x,y] \oplus [u,v]) \oplus [w,z]$

$= [x+u-2,\ y+v+3] \oplus [w,z]$

$= [x+u-2+w-2,\ y+v+3+z+3]$

$= [x+(u+w-2)-2,\ y+(v+z+3)+3]$

$= [x,y] \oplus ([u+w-2,\ v+z+3])$

$= [x,y] \oplus ([u,v] \oplus [w,z])$

$= v_1 \oplus (v_2 \oplus v_3)$

$\Rightarrow (v_1 \oplus v_2) + v_3 = v_1 \oplus (v_2 \oplus v_3)$

(5) Find Additive Identity or Zero vector in V is 0

$0 = 0 \odot [x,y] = [0x - 2(0) + 2,\ 0y + 3(0) - 3]$

$= [2,-3]$

$\Rightarrow$ Additive Identity is $[2,-3]$

$\Rightarrow \forall [x,y] \in V\ \exists$ Additive Identity $[2,-3]$ s.t

$[x,y] \oplus [2,-3] = [x+2-2,\ y-3+3]$

$= [x,y]$

(6) find the Additive Inverse for any element $[x,y] \in V\ \exists -[x,y]$ s.t

$-([x,y]) = (-1) \odot [x,y]$

$= [(-1)x - 2(-1) + 2,\ (-1)y + 3(-1) - 3]$

$= [-x+4,\ -y-6]$

$\Rightarrow [-x+4,\ -y-6]$ is additive Inverse

$\Rightarrow \forall [x,y] \in V\ \exists -([x,y])$ s.t

$[x,y] \oplus (-[x,y])$

$= [x,y] \oplus [-x+4,\ -y-6]$ $\quad (\because -([x,y]) = [-x+4,\ -y-6])$

$= [x,y] \oplus [-x+4,\ -y-6]$

$= [x-x+4-2,\ y-y-6+3]$

$= [2,-3]$ which is additive Identity in V

(7) Distributive Prop: $\forall a \in \mathbb{R}$ & $v_1 = [x,y]$, $v_2 = [w,z] \in V$

$a \odot (v_1 \oplus v_2) = a \odot ([x,y] \oplus [w,z])$

$= a \odot [x+w-2,\ y+z+3]$

$= [a(x+w-2) - 2a + 2,\ a(y+z+3) + 3a - 3]$

$= [ax + a\omega - 2a - 2a + 2,\ ay + az + 3a + 3a - 3]$

$= [ax - 2a + 2,\ ay + 3a - 3] \oplus [a\omega - 2a + 2,\ az + 3a - 3]$

$= a \odot [x, y] \oplus a \odot [\omega, z]$

$= (a \odot u_1) \oplus (a \odot u_2)$

$\Rightarrow a \odot (u_1 \oplus u_2) = (a \odot u_1) \oplus (a \odot u_2)$

(8) Distributive Property

$\forall\ a, b \in \mathbb{R}$ and $u = [x, y] \in V$

$(a+b) \odot u = (a+b) \odot [x, y]$

$= [(a+b)x - 2(a+b) + 2,\ (a+b)y + 3(a+b) - 3]$

$= [ax + bx - 2a - 2b + 2,\ ay + by + 3a + 3b - 3]$

$= [ax - 2a + 2 + bx - 2b,\ ay + 3a - 3 + by + 3b]$

$= [ax - 2a + 2,\ ay + 3a - 3] \oplus [bx - 2b + 2,\ by + 3b - 3]$

$= a \odot [x, y] \oplus b \odot [x, y]$

$= (a \odot u) \oplus (b \odot u)$

$\Rightarrow (a+b) \odot u = (a \odot u) \oplus (b \odot u)$

(9)

(9) Associative property of scalar multiplication $\forall\ a, b \in \mathbb{R}$ and $u = [x, y] \in V$

$(ab) \odot u = (ab) \odot [x, y]$

$= [(ab)x - 2(ab) + 2,\ (ab)y + 3(ab) - 3]$

$= [a(bx) - a(2b) + 2a - 2a + 2,\ a(by) + a(3b) - 3a + 3a - 3]$

$= [a(bx - 2b + 2) - 2a + 2,\ a(by + 3b - 3) + 3a - 3]$

$= a \odot [bx - 2b + 2,\ by + 3b - 3]$

$= a \odot [b \odot [x, y]]$

$= a \odot (b \odot u)$

$\Rightarrow (ab) \odot u = a \odot (b \odot u)$

(10) Identity under scalar multiplication

For each $u \in V$

$1 \odot u = 1 \odot [x, y]$

$= [1(x) - 2(1) + 2,\ 1(y) + 3(1) - 3]$

$= [x - 2 + 2,\ y + 3 - 3]$

$= [x, y]$

$\Rightarrow 1 \odot u = u$

$\Rightarrow V = \mathbb{R}^2$ is vector space.

Qn Prove that $\mathbb{R}^2$ is a vector space using operations ⊕ and ⊙ given by

~~$[x,y] \oplus [w,z]$~~

$[x,y] \oplus [w,z] = [x+w+1, y+z-2]$

And

$a \odot [x,y] = [ax+a-1, ay-2a+2]$

Also find the zero vector i.e. Additive Identity & Additive Inverse in V [Do yourself]

Qn Let V be a vector space over $\mathbb{R}$ then for any vector v in V and every non-zero real numbers a prove that $a v = 0$ iff $v = 0$

Proof Let $av = 0$ To show $v = 0$

$\because$ a is non-zero real no

$\Rightarrow a^{-1}$ exist

$av = 0$ —①

multiplying a^{-1} in ① both side

$a^{-1}(av) = a^{-1} 0 = 0$

$\Rightarrow (a^{-1}a)v = 0$

$1v = 0$

$v = 0$

Conversely, let $v = 0$

it is trivially, if $v = 0$ & $a \neq 0$ real no

$\Rightarrow a \cdot v = 0$ (H.P)

Qn Prove that the set of all scalar multiples of the vector $[1,3,4]$ in $\mathbb{R}^3$ forms a vector space with the usual operations addition and multiplication on 3-vectors [Do yourself]

Qn $V = \{A_{2\times 2} : trace(A) = 0\}$

then to show V is vector space over $\mathbb{R}$

Qn $V = \{(a_1, a_2) : a_1 a_2 = 1\}$ over $\mathbb{R}$ check it is vector space over $\mathbb{R}$

Qn $V = \{(a, 0) : a \in \mathbb{R}\}$ to show V is vector space over $\mathbb{R}$

Vector Space
Lecture - 5

Defn. Subspace: A subset W of a vector space V is called subspace of V if W is itself a vector space under the operation of addition and scalar multiplication defined on V

Let W be a non empty subset of a vector space V. Then W is a subspace of V iff W is closed under vector addition and scalar multiplication in V

Test for subspace
W is subspace of V
① $W \neq \phi$
② $\forall x, y \in W \Rightarrow x - y \in W$ or $x+y \in W$
③ $\alpha \in F \Rightarrow \alpha x \in W$

most important test for W is subspace

→ If W is subspace of V then then $0 \in W$
if $0 \notin W$ then W is not subspace

→ If $\alpha \in F$ & $x \in W$ then $\alpha x \in W$
If ~~for~~

If we find some $\alpha \in F$ s.t $\alpha x \notin W \Rightarrow W$ is not subspace

$F_1(F)$ is subspace of $F_2(F)$ iff
① $F' = F'$ ② $F_1 \subseteq F_2$

If any one of conditions is not hold [from ① and ②] then $F_1(F)$ is not subspace of $F_2(F)$

F(F) always form subspace

Q. Show that the set of vectors of the form $[0, a, 0, b]$ in $\mathbb{R}^4$ forms a subspace of $\mathbb{R}^4$ under the usual operations
$W = \{[0, a, 0, b] : a, b \in \mathbb{R}\}$

Sol^n For proving W is vector space to show
$\forall \alpha \in F$ & $\forall x \in W \Rightarrow \alpha x \in W$
& $\forall x, y \in W \Rightarrow x - y \in W$

let $\alpha \in \mathbb{R}$ & $x \in W \Rightarrow x = [0, a, 0, b]$
$x \in W$

$\alpha x = \alpha [0, a, 0, b]$
$= [0, \alpha a, 0, \alpha b]$
[$\because$ $a, b \in \mathbb{R}$
$\Rightarrow \alpha a, \alpha b \in \mathbb{R}$]
$= [0, \alpha a, 0, \alpha b] \in W$