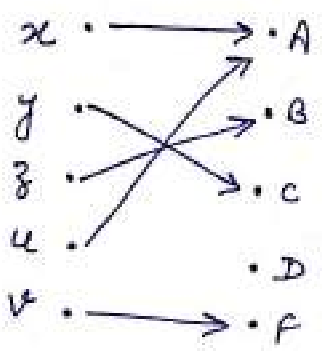
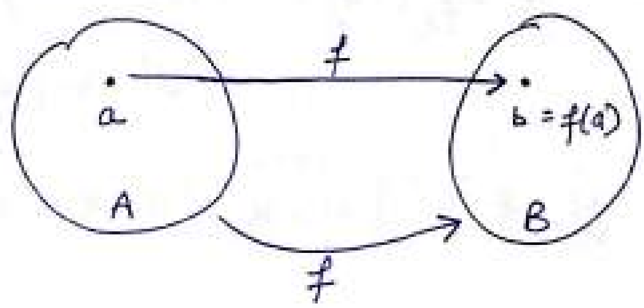
2.3. Functions (Rosen)

Functions :- Let A and B be non-empty sets. A function f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f: A \to B$.

functions also called mappings or transformations.

eg:- Assignment of Grades in a class.





If f is a function from A to B, A is domain of f, B is codomain of f.

If $f(a) = b$, 'b' is the image of a & a is a preimage of b.

Range of f is the set of all images of elements of A.

In above eg:- domain $= \{x, y, z, u, v\}$

codomain $= \{A, B, C, D, F\}$

range $= \{A, B, C, F\}$

Let f_1 & f_2 be functions from A to R. Then $f_1 + f_2$ & $f_1 f_2$ are also functions from A to R defined by :-

$$(f_1 + f_2)(x) = f_1(x) + f_2(x),$$

$$(f_1 f_2)(x) = f_1(x) \cdot f_2(x).$$

eg:- Let f_1 and f_2 be functions from R to R such that $f_1(x) = x^2$, $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$.

Solⁿ: $(f_1 + f_2)(x) = f_1(x) + f_2(x)$

$$= x^2 + (x - x^2) = x$$

$$(f_1 f_2)(x) = x^2(x - x^2) = x^3 - x^4.$$

Let f be a function from the set A to the set B and let S be a subset of A. The image of S under the function f is the subset of B that consists of the images of the elements of S. We denote image of S by $f(S)$, so,

$$f(S) = \{ t \mid \exists s \in S \ (t = f(s)) \}$$

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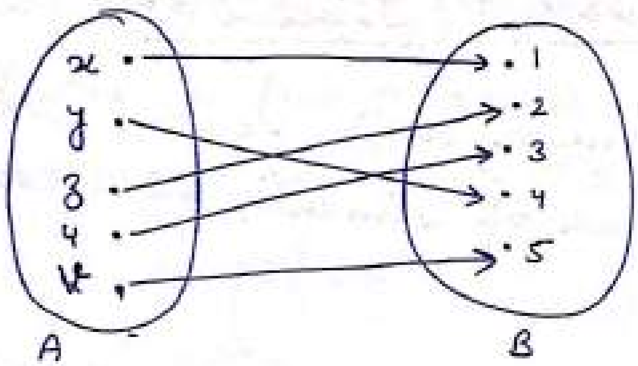
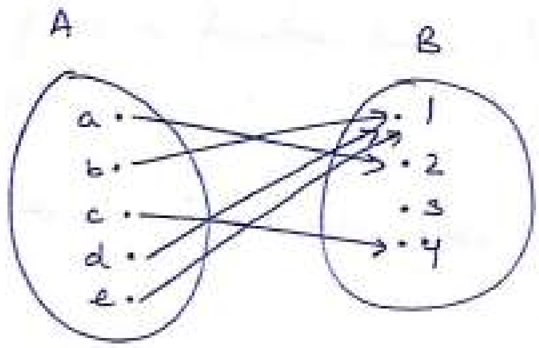


Let $S = \{z, 4, v\}$ ie $S \in A$. & $S \subseteq A$

ie image of $S = \{2, 3, 5\} \in B$ & image of $S \subseteq B$

ie $f(S) = \{2, 3, 5\}$.

eg:



$f(a) = 2$

$f(b) = 1$

$f(c) = 4$

$f(d) = 1$

$f(e) = 1$

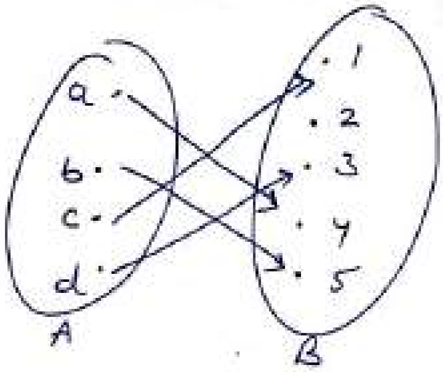
Let $S = \{b, c, d\} \subseteq A$.

then $f(S) = \{1, 4\}$

One-to-one functions (Injective f^n)

functions that never assign the same value to two different domain elements, are called one-to-one functions.

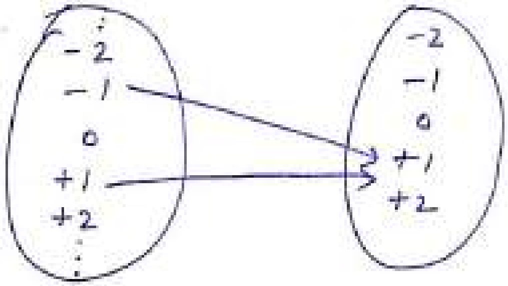
eg. Determine whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, $f(d) = 3$ is one-to-one



The function is one to one as each element in A is having a unique element in B.

eg:- Determine whether the f^n $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

sol^n.

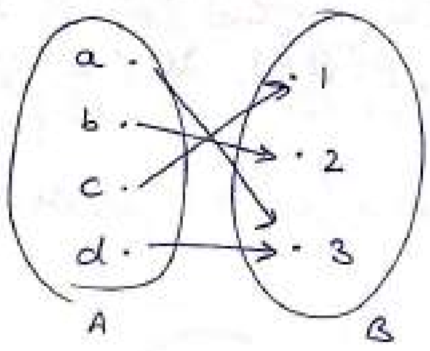


$f(1) = 1^2$ & $f(-1) = (-1)^2$

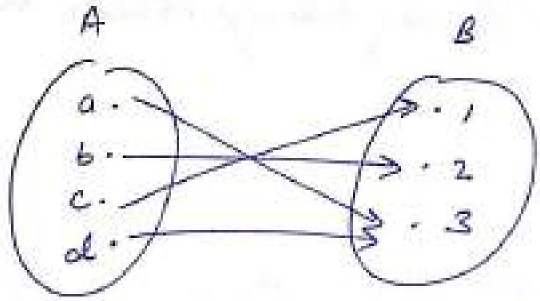
Both are equal to 1. Hence assignment is not unique. The function is not one-to-one f^n.

onto function (surjective)

A function f from A to B is called onto function, iff for every element $b \in B$ there is an element (atleast one) $a \in A$ with $f(a) = b$.



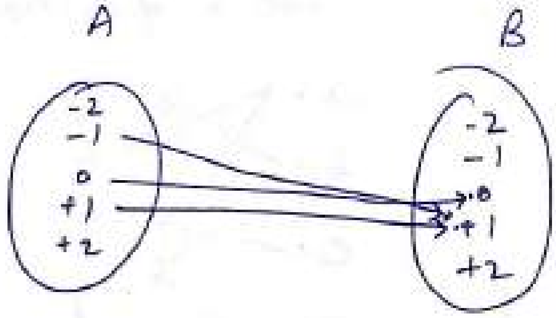
eg: let f be a function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, $f(d) = 3$. Is f onto f^n?



\# each element in B is assigned an element in A, Hence it is onto f^n.

eg: Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

sol^n: No,

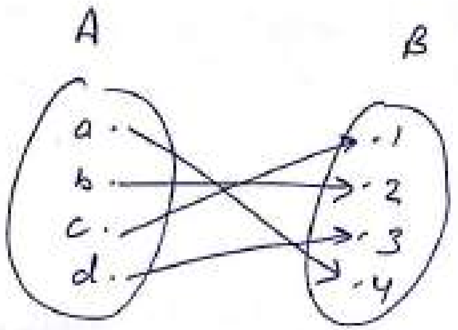


No mapping for -1, $+2$, etc.

ie no integer x with $x^2 = -1$

The function f is a <u>one-to-one correspondance</u> <u>or bijection</u>, if it is both one-to-one and onto.

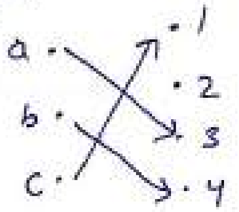
<u>eg</u>:- Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, $f(d) = 3$. Is f a bijection.



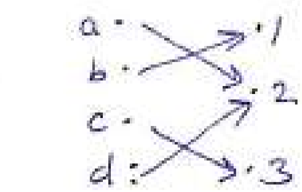
<u>one-to-one</u> :- Each element in A assigned a unique element in B

<u>onto</u> :- All 4 elements in B is having atleast one element in A.

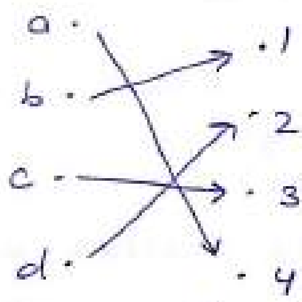
<u>egs</u>.



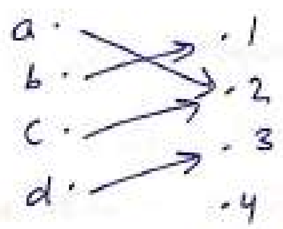
one-to-one, not onto



onto, ~~not~~ not one-to-one

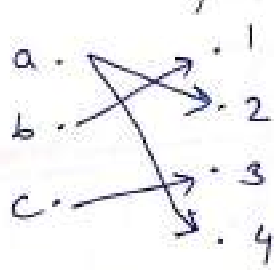


one to one & onto.



Neither one to one, nor onto

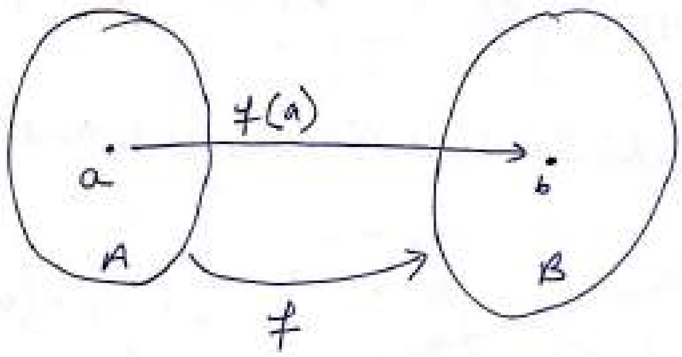
Not a function



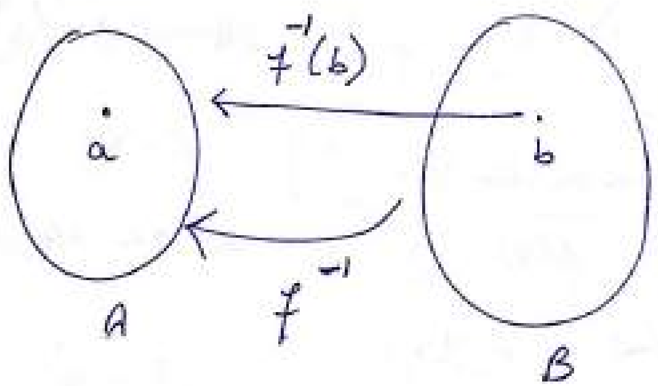
Inverse functions

Let f be a one-to-one correspondance from the set A to the set B. The inverse function of f is the function that assigns ~~to~~ an element b belonging to B the unique element ~~is~~ a in A such that $f(a) = b$. The inverse f^{-1} of f is denoted by f^{-1}.

Hence, $f^{-1}(b) = a$, when $f(a) = b$.



$f(a) = b$



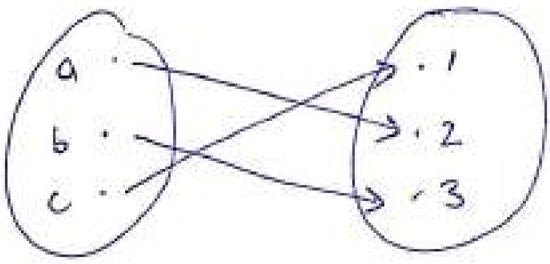
$f^{-1}(b) = a$

invertible :- one-to-one correspondance is called invertible because we can define an inverse of this f^n.

Not invertible :- A f^n is not invertible if it is not a one to one correspondance.

eg. Let f be a f^n from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, $f(c) = 1$. Is f invertible, if yes, find f^{-1}.

Sol^n: f is invertible since it is one-to-one correspondance.



$\therefore f^{-1}(1) = c, \quad f^{-1}(2) = a, \quad f^{-1}(3) = b$

steps to find inverse of a f^n :-

1). Replace the f^n $f(x)$ by y in the eq. describing the f^n
2). swap the variables x & y.
3) solve for y.
4). Replace y by $f^{-1}(x)$.

Q. Find inverse of $f(x) = \frac{x+1}{x}$.

Solⁿ.

$$f(x) = \frac{x+1}{x}$$

$$y = \frac{x+1}{x} \qquad [\text{step } 1].$$

$$x = \frac{y+1}{y} \qquad [\text{step } 2].$$

$$xy = y+1$$

$$xy - y = 1$$

$$y(-1+x) = 1$$

$$y = \frac{1}{-1+x} \qquad [\text{step } 3].$$

$$f^{-1}(x) = \frac{1}{x-1} \qquad [\text{step } 4].$$

Q. $f(x) = x^3 + 2$.

Solⁿ.

$$y = x^3 + 2$$

then $x = y^3 + 2$.

or $y^3 = \cancel{2-x}\ x-2$

$$y = (x-2)^{\frac{1}{3}}$$

$$f^{-1}(x) = (x-2)^{\frac{1}{3}}$$

Q. $f(x) = 5x - 7$, $\quad f^{-1}(x) = \frac{x+7}{5}$

Q. $f(x) = \frac{8}{9-3x}$ $\quad f^{-1}(x) = \frac{\left(\frac{8}{x}\right) - 9}{-3}$

Q $f(x) = \frac{7+4x}{6-5x}$, $f^{-1}(x) = \frac{6x-7}{4+5x}$

Q $f(x) = \cancel{\ } \sqrt[3]{x-2}$, $f^{-1}(x) = x^3+2$

Q $f(x) = \frac{10}{\sqrt[5]{7-3x}}$, $f^{-1}(x) = \frac{\left(\frac{10}{x}\right)^5 - 7}{-3}$

Q. $f(x) = 4e^{(6x+2)}$

$y = 4e^{6x+2}$

$x = 4e^{6y+2}$

~~Takin~~

$\frac{x}{4} = e^{6y+2}$

Taking natural log b.sides,

$\log_e\left(\frac{x}{4}\right) = \log_e e^{6y+2} = (6y+2)$

ie $\log_e\left(\frac{x}{4}\right) = 6y+2$

$y = \frac{\log_e\left(\frac{x}{4}\right) - 2}{6} = f^{-1}(x).$

Q. $f(x) = \log_e(2+4x)$

$y = \log_e(2+4x)$

$x = \log_e(2+4y)$

$e^x = e^{\log_e(2+4y)}$

$e^x = 2+4y$

$\frac{e^x-2}{4} = y$

$f^{-1}(x) = \frac{e^x-2}{4}$

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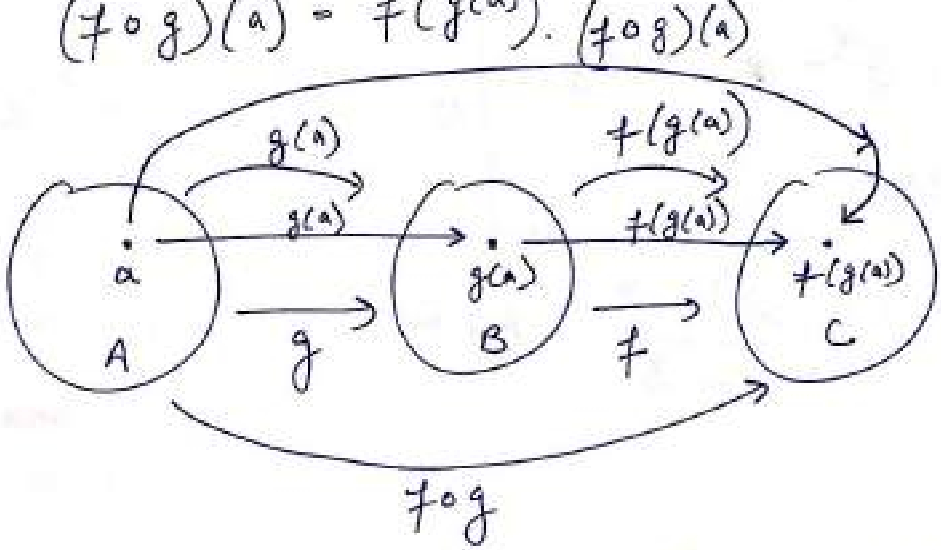
Composition of functions

Let g be a function from the set A to the set B
& let f be a f^n from the set B to the set C.
The composition of the functions f and g, denoted
by $f \circ g$, is defined by

$$(f \circ g)(a) = f(g(a)).$$



we first apply function g to a obtain $g(a)$ & then we apply the function f to the result $g(a)$ to obtain

$(f \circ g)(a) = f(g(a))$.

composition $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f.

eg. Let g be a f^n from the set $\{a,b,c\}$ to itself such that $g(a) = b$, $g(b) = c$, $g(c) = a$.

Let f be a f^n from the set $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f(a) = 3$, $f(b) = 2$, $f(c) = 1$. find $f \circ g$ & $g \circ f$.

Sol^n. $f \circ g$.

composition $f \circ g$ is defined by $(f \circ g)(a) = \cancel{f(g(a))}$.

ie $\cancel{f(b)} = f(g(a)) = f(b) = 2$

$f(g(b)) = f(c) = 1$

$f(g(c)) = f(a) = 3$.

composition $g \circ f$ is defined by $(g \circ f)(a)$.

ie $g(f(a)) = g(3) = N.D$

$g(f(b)) = g(2) = N.D$

$g(f(c)) = g(1) = N.D.$

$\therefore g \circ f$ does not exist.

or we can say range of f is not a subset of the domain of g.

eg. Let f and g be functions from the set of integers to set of integers.

$f(x) = 2x+3$, $g(x) = 3x+2$.

find $f \circ g$?

$(f \circ g)(x) = f(g(x)) = f(3x+2) = 2(3x+2)+3 = 6x+7$

find $g \circ f$? $(g \circ f)(x) = g(f(x)) = g(2x+3) = 3(2x+3)+2 = 6x+11$.

Commutative law does not hold for composition of functions ie $f \circ g \neq g \circ f$.

If $f(a) = b$, then $f^{-1}(b) = a$

If $f^{-1}(b) = a$, then $f(a) = b$.

Hence

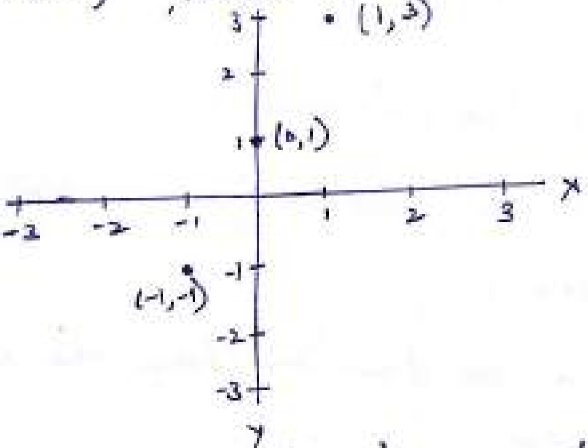
$$(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a$$

& $$(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b.$$

<u>Graphs of functions</u>

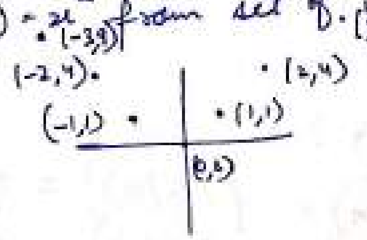
<u>Q.</u> Display the graph of function $f(x) = 2x+1$ from the set of integers to the set of integers.

<u>Solⁿ.</u> The graph of f is the set of ordered pairs of the form $(x, 2x+1)$, where x is an integer.



<u>Q.</u> Display the graph of f^n $f(x) = x^2$ from set of integers to set of integers

<u>Solⁿ.</u> ordered pair (x, x^2)



Floor & ceiling functions

floor :- $\lfloor \ \rfloor$ eg :- $\lfloor 2 \rfloor = 2$, $\lfloor 2.5 \rfloor = 2$, $\lfloor 3.5 \rfloor = 3$

ceiling :- $\lceil \ \rceil$. eg :- $\lceil 2 \rceil = 2$, $\lceil 2.5 \rceil = 3$, $\lceil 3.5 \rceil = 4$.

Properties of floor and ceiling functions (n is an integer)

1a) $\lfloor x \rfloor = n$ iff $n \leq x < n+1$

eg $\lfloor 2.5 \rfloor = 2$ iff $2 \leq 2.5 < 3$ (True).

1b). $\lceil x \rceil = n$ iff $n-1 < x \leq n$

eg $\lceil 2.5 \rceil = 3$ iff $2 < 2.5 \leq 3$

eg $\lceil 2 \rceil = 2$ iff $1 < 2 \leq 2$

1c) $\lfloor x \rfloor = n$ iff $x-1 < n \leq x$

$\lfloor 2.5 \rfloor = 2$ iff $1.5 < 2 \leq 2.5$

$\lfloor 2 \rfloor = 2$ iff $1 < 2 \leq 2$

1d). $\lceil x \rceil = n$ iff $x \leq n < x+1$

eg $\lceil 3.5 \rceil = 3$ iff $3.5 \leq 3 < 4.5$

eg. $\lceil 3 \rceil = 3$ iff $3 \leq 3 < 4$

(2). $x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$

$$2.5 < \lfloor 3.5 \rfloor \leq 3.5 \leq \lceil 3.5 \rceil < 4.5$$

(3a). $\lfloor -x \rfloor = -\lceil x \rceil$

$\lfloor -2.5 \rfloor = -\lceil 2.5 \rceil$

$\Rightarrow -3 = -3.$

(3b) $\lceil -x \rceil = -\lfloor x \rfloor$

$\Rightarrow \lceil -3.5 \rceil = -\lfloor 3.5 \rfloor$

$\Rightarrow -3 = -3$

(4a) $\lfloor x+n \rfloor = \lfloor x \rfloor + n$

eg $\lfloor 1.5+1 \rfloor = \lfloor 1.5 \rfloor + 1$

$\Rightarrow \lfloor 2.5 \rfloor = 1+1$

$\Rightarrow 2 = 2$

(4b) $\lceil x+n \rceil = \lceil x \rceil + n$

$\lceil 2.5+1 \rceil = \lceil 2.5 \rceil + 1$

$\Rightarrow 4 = 3+1$

$\Rightarrow 4 = 4.$

Q **Prove.** $\lfloor x+n \rfloor = \lfloor x \rfloor + n$
where 'n' is an integer & 'x' a real no.

Proof. suppose that $\lfloor x \rfloor = m$, m is a +tive integer.
using 1ⓐ, it follows that,

$$m \leq x < m+1$$

Adding n to b.s of this inequality.

$$m+n \leq x+n < m+n+1$$

using 1ⓐ again, [1ⓐ $\Rightarrow$ $\lfloor x \rfloor = n$ iff $n \leq x < n+1$]

$$\lfloor x+n \rfloor = m+n$$

$$\lfloor x+n \rfloor = \lfloor x \rfloor + n \quad \text{[supposition used here]}$$

$$= \text{R.H.S.}$$

Hence Proved.

Approach for floor functions is to let,
$x = n + \varepsilon$, where $n = \lfloor x \rfloor$ is an integer &
(eg. $3.5 = 3 + .5$) & ε, the fractional part of x, where
$(0 \leq \varepsilon < 1)$

Approach for ceiling f^n is to let

$x = n - \varepsilon$, where $n = \lceil x \rceil$ is an integer
(eg $3.5 = 4 - 0.5$) & ε the fractional part of x,
where $(0 \leq \varepsilon < 1)$.

eg. Prove or disprove that $\lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil$ for all real nos x & y.

Solⁿ. Proved for values $x = 1.5$, $y = 1.6$.

$$\lceil 1.5 + 1.6 \rceil = \lceil 1.5 \rceil + \lceil 1.6 \rceil$$

$$\Rightarrow \lceil 3.1 \rceil = 2 + 2$$

$$= 4 = 4 \quad (L.H.S = R.H.S)$$

disproved using counter example,

$$x = 0.5, \quad y = 0.5$$

$$\lceil 0.5 + 0.5 \rceil \neq \lceil 0.5 \rceil + \lceil 0.5 \rceil$$

$$\Rightarrow \lceil 1.0 \rceil \neq \lceil 1 \rceil + \lceil 1 \rceil$$

$$= 1 \neq 2. \quad (LHS \neq R.H.S)$$

Hence the equality is disproved.

Q. Show that if x is a real number, then,

$$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1.$$

Solⁿ. write a real nos. x as $\lfloor x \rfloor + \varepsilon$ $[2.5 = 2 + 0.5]$

where ε is a real no with $0 \leq \varepsilon < 1$ — (1)

ie $x = \lfloor x \rfloor + \varepsilon$

or

$\varepsilon = x - \lfloor x \rfloor$

$\therefore$ we can write $0 \leq x - \lfloor x \rfloor < 1$ $[0.5 < 1]$

or

$0 \leq x - 1 < \lfloor x \rfloor$ $[1.5 < 2]$

(first inequality Proved).

Also, if $x = \lfloor x \rfloor + \varepsilon$

then $x \geq \lfloor x \rfloor$ [second inequality Proved].

Now, for the other two inequalities

$x = \lceil x \rceil - \varepsilon'$, $[2.5 = 3 - 0.5]$

where $\varepsilon' = \lceil x \rceil - x$

ie $0 \le \varepsilon' < 1$

thus $0 \le \lceil x \rceil - x < 1$

or $0 \le \lceil x \rceil < 1 + x$ (fourth inequality follows).

Now, if $x = \lceil x \rceil - \varepsilon'$

then $x \leqslant \lceil x \rceil$ (third inequality follows.)

Q. show that if x is a real no. & n is an integer, then,

(a) $x < n$ iff $\lfloor x \rfloor < n$

sol^n. suppose, $x < n$

then $\lfloor x \rfloor + \varepsilon < n$ $[\because x = \lfloor x \rfloor + \varepsilon]$

or $\lfloor x \rfloor < n$

which is given, hence Proved.

(b) $n < x$ iff $n < \lceil x \rceil$

suppose $n < x$ (Assumption)

$n < \lceil x \rceil - \varepsilon$ $[x = \lceil x \rceil - \varepsilon]$

or $n + \varepsilon < \lceil x \rceil$

or $n < \lceil x \rceil$

which is given, hence Proved.

Q. Prove that if n is an integer, then $\lfloor \frac{n}{2} \rfloor = \frac{n}{2}$ if n is even,

and $\frac{(n-1)}{2}$ if n is odd.

Solⁿ.

If n is even, then $n = 2K$ for some integer K

thus $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{2K}{2} \rfloor = \lfloor K \rfloor = K = \frac{n}{2}$

If n is odd, then $n = 2K+1$, for some integer K,

Thus, $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{2K+1}{2} \rfloor = \lfloor K + \frac{1}{2} \rfloor = K = \frac{n-1}{2}$

Q. Show that if x is a real number & n is an integer, then

a) $x \leq n$ iff $\lceil x \rceil \leq n$.

b) $n \leq x$ iff $n \leq \lfloor x \rfloor$

Q. Show that if x is a real no and m is an integer, then, $\lceil x+m \rceil = \lceil x \rceil + m$.

Solⁿ. Suppose that,

$$\lceil x \rceil = n \quad \text{where } n \text{ is a +tive integer,}$$

By Prop. (1b), we know that,

(1b) $\boxed{\lceil x \rceil = n \text{ iff } n-1 < x \leq n}$

Add m b.sides,

$$m+n-1 < x+m \leq m+n$$

Again using (1b).

$$\lceil x+m \rceil = m+n$$

$$= m + \lceil x \rceil = \text{R.H.S.}$$

Q. Prove or disprove :-

① $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$ for all real numbers x.

solⁿ. True, because $\lfloor x \rfloor$ itself is an integer.
Hence $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$.

② $\lfloor 2x \rfloor = 2\lfloor x \rfloor$, x is real no. (Proved for $x = 0.1$)

solⁿ. False, counterexample is :-
consider $x = 0.5$

$\lfloor 2 * 0.5 \rfloor \neq 2 \lfloor 0.5 \rfloor$

$\lfloor 1 \rfloor \neq 2.0$

$1 \neq 0$ Hence False.

③ $\lceil x \rceil + \lceil y \rceil - \lceil x+y \rceil = 0$ or 1 whenever x & y are real nos.

solⁿ # True, if x or y is an integer, then by Property

$\lceil x + n \rceil = \lceil x \rceil + n$ (n is an integer, x is real)

the difference is 0.

If neither 'x' nor 'y' is an integer, then

$x = n + \varepsilon$ & $y = m + \delta$

where n, m are integers & ε, δ are real nos < 1.

Then, $m + n < x + y < m + n + 2$,

so $\lceil x+y \rceil = m + n + 1$

or

$\boxed{\lceil x + y \rceil \leq m + n + 2}$ (using 1b).

[⑯. $\lceil x \rceil = n$ iff $n - 1 < x \leq n$]

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Therefore, the given exp. can be written as,

$$\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil$$

$$\rightarrow (n+1) + (m+1) - (m+n+1) = 1$$

$$\boxed{\left(\because x = n + \varepsilon \text{ or } \lceil x \rceil = n+1 \quad \text{and} \quad y = m + \delta \text{ or } \lceil y \rceil = m+1 \right)}$$

or

$$\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil$$

$$\Rightarrow (n+1) + (m+1) - (m+n+2) = 0$$

Hence Proved.

Q. $\lceil xy \rceil = \lceil x \rceil \lceil y \rceil$ for all real no x & y.

Ans. False, Prove using counterexample, (Proved for $x = 0.5, y = 0.5$)

consider $x = 0.25$, $y = 1.5$

$$\lceil xy \rceil = \lceil 0.25 * 1.5 \rceil = \lceil 0.375 \rceil = 1$$

$$\lceil x \rceil \lceil y \rceil = \lceil 0.25 \rceil \lceil 1.5 \rceil = 1 . 2 = 2 \neq \text{L.H.S.}$$

Hence Disproved.

Q. $\lceil \frac{x}{2} \rceil = \lfloor \frac{x+1}{2} \rfloor$ for all real nos x. (Proved for $x = 2.0$)

Ans. False, Prove using counter example,

consider $x = 0.5$,

$$\lceil \frac{0.5}{2} \rceil = \lceil 0.25 \rceil = 1$$

$$\lfloor \frac{0.5+1}{2} \rfloor = \lfloor \frac{1.5}{2} \rfloor = \lfloor 0.75 \rfloor = 0 \neq \text{LHS.}$$ Hence Disproved

29. Prove that if x is a real number, then $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$