$$A = U_s(\omega,0) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} u(x,0)\sin\omega x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^1 1\cdot\sin\omega x\,dx + \int_1^{\infty} 0\,\sin\omega x\,dx\right]$$

$$= -\sqrt{\frac{2}{\pi}}\left[\frac{\cos\omega x}{\omega}\right]_0^1 + 0$$

$$\text{ie } A = \sqrt{\frac{2}{\pi}}\;\frac{1-\cos\omega}{\omega} \quad \text{--- (6)}$$

Now eqn (4) becomes

$$u_s = \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos\omega}{\omega}\right)e^{-\omega^2 t} \quad \text{--- (7)}$$

Applying inverse Fourier sine transform, we get

$$u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} u_s \sin\omega x\,d\omega$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos\omega}{\omega}\right)\sin\omega x\,d\omega$$

$$= \frac{2}{\pi}\int_0^{\infty}\left(\frac{1-\cos\omega}{\omega}\right)\sin\omega x\,d\omega$$

This is the required solution

Wave Equation (One dimensional)

Example:- Use the method of Fourier transform to determine the displacement $y(x,t)$ of an infinite string, given that the string is initially at rest and the initial displacement is $f(x)$, $-\infty<x<\infty$. Show that the solution can also be put in the form

$$y(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$$

Solution:- Displacement $y(x,t)$ of any point of an infinite string is governed by the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \quad \text{for } -\infty<x<\infty,\ t>0 \quad \text{--- (1)}$$

with the conditions (i) $y(x,0) = f(x)$

(ii) $\frac{\partial y}{\partial t} = 0$ at $t=0$ (since the string is initially at rest)

From eqn (1)

$$F.T.\left[\frac{\partial^2 y}{\partial t^2}\right] = c^2 F.T.\left[\frac{\partial^2 y}{\partial x^2}\right]$$

$$\text{ie } \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^2 y}{\partial t^2}e^{-i\omega x}dx = c^2\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^2 y}{\partial x^2}e^{-i\omega x}dx$$

$$\text{ie } \frac{\partial^2}{\partial t^2}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y\,e^{-i\omega x}dx\right\} = c^2(i\omega)^2 Y(\omega,t)$$

or $\dfrac{\partial^2 Y(\omega,t)}{\partial t^2} + v^2\omega^2 Y(\omega,t) = 0$ — ② $\quad Y(\omega,t) = F.T[y(x,t)]$

where $Y(\omega,t) = F.T(y(x,t))$, The solution of eqn ② is

$Y(\omega,t) = A\cos v\omega t + B\sin v\omega t$ — ③

The boundary condition is $\dfrac{\partial y}{\partial t} = 0$ at $t=0$ implies

$$F.T\left[\frac{\partial y}{\partial t}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \frac{\partial y}{\partial t} e^{-i\omega x} dx = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} y e^{-i\omega x} dx\right\} = 0$$

ie $\dfrac{\partial}{\partial t} Y(\omega,t) = 0$ at $t=0$ — ④

Diff. ③ and

$$\frac{dY(\omega,t)}{dt} = -Av\omega\sin v\omega t + Bv\omega\cos v\omega t$$

Now $\left.\dfrac{dY}{dt}\right|_{t=0} = Bv\omega\cos v\omega\cdot 0 = 0$ from ④

$= Bv\omega = 0$

$\therefore B = 0.$

Hence $Y(\omega,t) = A\cos v\omega t$ — ⑤

Now from initial condition $y(x,0) = f(x)$

$\therefore F.T[y(x,0)] = F.T[f(x)]$

$$Y(\omega,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x) e^{-i\omega x} dx = g(\omega). \quad \text{— ⑥}$$

From ⑤ $\quad Y(\omega,0) = A = g(\omega)$ ⑦ from ⑥.

Hence from ⑤

$$Y(\omega,t) = F.T[y(x,t)] = g(\omega)\cos v\omega t \quad \text{— ⑧}$$

Taking Inverse fourier transform, we get

$$y(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} Y(\omega,t) e^{i\omega x} d\omega.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(\omega)\cos v\omega t \, e^{i\omega x} d\omega$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(\omega)\left[\frac{e^{iv\omega t} + e^{-iv\omega t}}{2}\right] d\omega \cdot e^{i\omega x}$$

$$= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(\omega)\left[e^{i\omega(x+vt)} + e^{i\omega(x-vt)}\right] d\omega.$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(\omega) e^{i\omega(x+vt)} d\omega + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(\omega) e^{i\omega(x-vt)} d\omega\right]$$

$= \dfrac{1}{2}\left[f(x+vt) + f(x-vt)\right]$ — which is the req. soln → wave function.

Parseval's identity for Fourier transform. Rayleigh's Theorem or Plancherel's theorem.

If $f(\omega)$ and $g(\omega)$ are the complex Fourier transforms of $F(x)$ and $G(x)$ respectively, then

(i) $$\int_{-\infty}^{+\infty} f(\omega)\overline{g(\omega)}\,d\omega = \int_{-\infty}^{+\infty} F(x)\overline{G(x)}\,dx$$

(ii) $$\int_{-\infty}^{+\infty} |f(\omega)|^2\,d\omega = \int_{-\infty}^{+\infty}|F(x)|^2\,dx$$

where bar signifies the complex conjugate.

Proof :- (i) Using the inversion formula for Fourier transform

$$G(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(\omega)e^{i\omega x}\,d\omega \quad -①$$

Taking complex conjugates on both sides, (1) gives

$$\overline{G(x)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\overline{g(\omega)}\,e^{-i\omega x}\,d\omega \quad -②$$

$$\therefore \int_{-\infty}^{+\infty} F(x)\overline{G(x)}\,dx = \int_{-\infty}^{+\infty} F(x)\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\overline{g(\omega)}\,e^{-i\omega x}\,d\omega\right\}dx$$

from ②

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\overline{g(\omega)}\left\{\int_{-\infty}^{+\infty} F(x)e^{-i\omega x}\,dx\right\}d\omega$$

$$= \int_{-\infty}^{+\infty}\overline{g(\omega)}f(\omega)\,d\omega$$

(ii) Taking $G(x) = F(x)$ in part (i), we obtain

$$\int_{-\infty}^{\infty} f(\omega)\overline{f(\omega)}\,d\omega = \int_{-\infty}^{\infty} F(x)\overline{F(x)}\,dx$$

$$\Leftrightarrow \int_{-\infty}^{+\infty}|f(\omega)|^2\,d\omega = \int_{-\infty}^{+\infty}|F(x)|^2\,dx.$$

Parseval's identities for Fourier cosine and sine transforms :-

(i) $$\int_0^{\infty} f_c(\omega)\,g_c(\omega)\,d\omega = \int_0^{\infty} F(x)\,G(x)\,dx$$

Prove $f * g = g * f$

$$f * g = \frac{1}{\sqrt{2\pi}} \int f(t') g(t-t')\, dt'$$

Put $y = t - t' \Rightarrow t' = t - y$

$dy = dt'$

$$f * g = \frac{1}{\sqrt{2\pi}} \int f(t-y)\, g(y)\, dy$$

$$= g * f$$

Prove $f * (g + h) = f * g + f * h$

$$f * (g+h) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t') \left[g(t-t') + h(t-t') \right] dt'$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t') g(t-t')\, dt' + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t') h(t-t')\, dt'$$

$$= f * g + f * h$$

Prove $(f * g) * h = f * (g * h)$

L.H.S

$$\left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t') g(t-t')\, dt' \right\} * h$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (f * g)\, h(t-t')\, dt'$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(t'') g(t'-t'') h(t-t')\, dt'\, dt''$$

Proof :- From Parseval's identity

$$\int_{-\infty}^{+\infty} F(x)\overline{G(x)}dx = \int_{-\infty}^{+\infty} f(\omega)\overline{g(\omega)}d\omega$$

For even function $F(x)$ and $\overline{G(x)}$

$$\int_{-\infty}^{+\infty} F(x)\overline{G(x)}\,dx = 2\int_0^{\infty} F(x)G(x)dx \quad \text{since } \overline{G(x)} = G(x) \text{ for even functions.}$$

Also if $F(x)$ is even $f(\omega)$ is also even and we write $f(\omega) = f_c(\omega)$ and if $G(x)$ is even $g(\omega)$ is also even and so $\overline{g(\omega)} = g(\omega) = g_c(\omega)$

and $\int_{-\infty}^{+\infty} f(\omega)\overline{g}(\omega)d\omega = 2\int_0^{\infty} f_c(\omega)g_c(\omega)d\omega$

$$\therefore \int_0^{\infty} F(x)G(x)dx = \int_0^{\infty} f_c(\omega)g_c(\omega)d\omega$$

Similarly other results that follows from Parseval's identity are

$$\int_0^{\infty} f_s(\omega)g_s(\omega)d\omega = \int_0^{\infty} F(x)G(x)dx$$

$$\int_0^{\infty} [f_c(\omega)]^2 d\omega = \int_0^{\infty} [F(x)]^2 dx$$

and $\int_0^{\infty} [f_s(\omega)]^2 d\omega = \int_0^{\infty} [F(x)]^2 dx$

Prove that $F.T[f(x)] = \frac{1}{\sqrt{2\pi}}\frac{1}{\omega^3}(4\sin\omega - 4\omega\cos\omega)$

$$\text{where } f(x) = 1-x^2 \quad |x| \leq 1$$
$$= 0 \quad |x| > 1$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{(a^2+b^2)}(a\cos bx + b\sin bx)$$

The Integral representation of the Delta function.

From Fourier integral theorem,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)\,dt \int_{-\infty}^{+\infty} e^{\pm iu(x-t)}\,du \quad —①$$

This can be compared with the following definition of the Dirac delta function

$$f(x) = \int_{-\infty}^{+\infty} f(t)\,\delta(x-t)\,dt \quad —②$$

Comparing ① and ②

$$\delta(x-t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\pm iu(x-t)}\,du \quad —③$$

We can also write

$$\delta(\omega-\omega') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\pm i(\omega-\omega')t}\,dt \quad —④$$

Now if we consider the function

$$E(t) = Ae^{i\omega_0 t} \quad -\infty < t < +\infty \quad —⑤$$

then its fourier transform is

$$f(\omega) = \frac{A}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\omega_0 t}\,e^{-i\omega t}\,dt$$

$$= \frac{A}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i(\omega_0-\omega)t}\,dt$$

$$= \frac{A}{\sqrt{2\pi}} \times \frac{\sqrt{2\pi}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i(\omega_0-\omega)t}\,dt$$

$$= A\sqrt{2\pi}\left\{\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(\omega_0-\omega)t}\,dt\right.$$

$$f(\omega) = A\sqrt{2\pi}\,\delta(\omega-\omega_0) \quad —⑥$$

⇒ eqn ⑤ describes a perfectly monochromatic wave, therefore, its frequency spectrum must contain only one frequency (ω_0) which is described by the Dirac Delta function.

Q.1. Find Fourier transform of

$$f(x) = \begin{cases} x, & |x| \le a \\ 0, & |x| > a \end{cases}$$

Soln. $F\{f(x)\} = g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\omega x} f(x)\, dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-a} e^{-i\omega x} f(x)\, dx + \int_{-a}^{a} e^{-i\omega x} f(x)\, dx + \int_{a}^{\infty} e^{-i\omega x} f(x)\, dx.$$

$$= \int_{-a}^{a} e^{-i\omega x} x\, dx$$

$$= x \frac{e^{-i\omega x}}{(-i\omega)} \Big|_{-a}^{a} - \int_{-a}^{a} 1 \cdot \frac{e^{-i\omega x}}{(-i\omega)}\, dx$$

$$= \frac{1}{(-i\omega)}\left[a e^{-i\omega a} + a e^{i\omega a}\right] + \frac{1}{i\omega} \int_{-a}^{a} e^{-i\omega x}\, dx.$$

$$= \frac{a}{(-i\omega)}\left[e^{-i\omega a} + e^{i\omega a}\right] + \frac{1}{i\omega} \frac{e^{-i\omega x}}{-i\omega}\Big|_{-a}^{a}$$

$$= \frac{2a}{(-i\omega)} \cos\omega a + \frac{1}{(i\omega)(-i\omega)}\left[e^{-i\omega a} - e^{i\omega a}\right]$$

$$= \frac{2a}{(-i\omega)} \cos\omega a + \frac{2}{i\omega^2}\left[\frac{e^{i\omega a} - e^{-i\omega a}}{2i}\right]$$

$$= \frac{2a}{-i\omega} \cos\omega a + \frac{2\sin\omega a}{i\omega^2}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2i}{\omega^2}\left[a\omega\cos\omega a - \sin\omega a\right] = \sqrt{\frac{2}{\pi}} \frac{i}{\omega^2}\left[a\omega\cos\omega a - \sin\omega a\right]$$

Q.2. Find the complex Fourier transform of

$$f(x) = e^{-a|x|}$$

where $a > 0$ and x belongs to $(-\infty, \infty)$

Soln. $g(\omega) = \int_{-\infty}^{+\infty} f(x) e^{-i\omega x}\, dx = \int_{-\infty}^{\infty} e^{-i\omega x} e^{-a|x|}\, dx$

$$= \int_{-\infty}^{0} e^{-i\omega x} e^{ax}\, dx + \int_{0}^{\infty} e^{-i\omega x} e^{-ax}\, dx$$

$$= \int_{-\infty}^{0} e^{x(a-i\omega)}\,dx + \int_{0}^{\infty} e^{-x(a+i\omega)}\,dx .$$

$$= \left.\frac{e^{x(a-i\omega)}}{a-i\omega}\right|_{-\infty}^{0} + \left.\frac{e^{-x(a+i\omega)}}{-(a+i\omega)}\right|_{0}^{\infty}$$

$$= \frac{1}{a-i\omega} + \frac{1}{a+i\omega}$$

$$= \frac{2a}{a^2+\omega^2}$$

LAPLACE'S TRANSFORM

The knowledge of Laplace transforms has become an essential part of mathematical background required of engineers and scientists. This is because the transform methods provide an easy and effective means for the solution of many problems arising in engineering.

The method of Laplace transform has the advantage of directly giving the solution of differential equations with given boundary values without the necessity of first finding the general solution and then evaluating from it the arbitrary constant. Moreover the ready tables of Laplace transforms reduce the problem of solving differential eqn to mere algebraic manipulation.

Definition:-

Let $F(t)$ be a function of t defined for all positive values of t. Then the Laplace transform of $F(t)$, denoted by $L\{F(t)\}$, is defined by

$$L\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\,dt \quad \text{---} (1)$$

Here the operator L is called the Laplace transformation operator. The parameter s may be real or complex number, but generally it is taken to be a real positive number.

The Laplace transform of a function $F(t)$ exists only if the function satisfies the following conditions.

(1) The function $F(t)$ should be an arbitrary piecewise continuous function in every finite interval and that $F(t)=0$ for all negative values of t.

(2) The function $F(t)$ should be of exponential order.

Properties of Laplace Transforms:-

1) Linearity Property:- If a_1 and a_2 are constants and the Laplace transforms of $F_1(t)$ and $F_2(t)$ are $f_1(s)$ and $f_2(s)$ resp., the Laplace transform of $a_1F_1(t)+a_2F_2(t)$ is given by $a_1f_1(s)+a_2f_2(s)$, i.e.

$$L\{a_1F_1(t)+a_2F_2(t)\} = a_1 L\{F_1(t)\} + a_2 L\{F_2(t)\}$$

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Proof:- Let $L\{F_1(t)\} = f_1(s) = \int_0^\infty e^{-st} F_1(t)\,dt$

and $L\{F_2(t)\} = f_2(s) = \int_0^\infty e^{-st} F_2(t)\,dt$

$\therefore L\{a_1F_1(t)+a_2F_2(t)\} = \int_0^\infty e^{-st}\left[a_1F_1(t)+a_2F_2(t)\right]dt$

$= a_1\int_0^\infty e^{-st}F_1(t)\,dt + a_2\int_0^\infty e^{-st}F_2(t)\,dt$

$= a_1 L\{F_1(t)\} + a_2 L\{F_2(t)\}$

Generalizing this result

$$L\left\{\sum_{m=1}^{n} a_m F_m(t)\right\} = \sum_{m=1}^{\infty} a_m L\{F_m(t)\}$$

2. Change of Scale Property :- If $f(s)$ is the Laplace transform of $F(t)$, the Laplace transform of $F(at)$ is $\frac{1}{a}f\left(\frac{s}{a}\right)$

Proof:- We have $L\{F(t)\} = f(s) = \int_0^\infty e^{-st}F(t)\,dt$

Therefore $L\{F(at)\} = \int_0^\infty F(at)\,e^{-st}\,dt$

Subs. $at = u$, i.e. $dt = \frac{du}{a}$, we get

$L\{F(at)\} = \int_0^\infty F(u)\,e^{-(s/a)u}\,\frac{du}{a}$

$= \frac{1}{a}\int_0^\infty F(u)\,e^{-(s/a)u}\,du = \frac{1}{a}\,f\left(\frac{s}{a}\right)$ — (2)

3. First Translation (Shifting) Property :- If $f(s)$ is Laplace transform of $F(t)$, then that of $e^{at}F(t)$ will be $f(s-a)$

Proof:- Let $L\{F(t)\} = f(s) = \int_0^\infty e^{-st}F(t)\,dt$

Then $L\{e^{at}F(t)\} = \int_0^\infty e^{-st}\left[e^{at}F(t)\right]dt$

$= \int_0^\infty F(t)\,e^{-(s-a)t}\,dt = f(s-a)$ — (3a)

Similarly, we may show $f(s+a) = L\{e^{-at}F(t)\}$ — (3b)

(4) Second Translation Property (Heaviside Shifting Theorem):- If $L\{F(t)\} = f(s)$ and $G(t) = \begin{cases} 0 & \text{if } t < a \\ F(t-a) & \text{if } t > a, \end{cases}$

then $L\{G(t)\} = e^{-as}f(s)$

Proof:- $L\{G(t)\} = \int_0^\infty e^{-st}G(t)\,dt = \int_0^a e^{-st}G(t)\,dt + \int_a^\infty e^{-st}G(t)\,dt$

$$= \int_a^\infty e^{-st} F(t-a)\, dt$$

Subs $t - a = p$, i.e. $dt = dp$, we get

$$L\{G(t)\} = \int_0^\infty e^{-s(p+a)} F(p)\, dp = e^{-as} \int_0^\infty e^{-sp} F(p)\, dp$$

$$L\{G(t)\} = e^{-as} f(s) \quad \text{—— (4a)}$$

This result may be expressed as

$$L\{F(t-a)\, U(t-a)\} = e^{-sa} f(s)$$

where $U(t-a) = \begin{cases} 1 & \text{for } t > a \\ 0 & \text{for } t < a \end{cases}$ is called Heaviside unit step input function

(Continued on page (19))

(5) Derivative of Laplace Transform :– If $f(s)$ is Laplace transform of $F(t)$, then $f'(s) = \frac{df}{ds} = L\{-tF(t)\}$ and in general $L\{t^n F(t)\} = (-1)^n f^n(s) = (-1)^n \frac{d^n f(s)}{ds^n}$

Proof :- We have $f(s) = \int_0^\infty e^{-st} F(t)\, dt$.

Differentiating both sides w.r.t s, we get

$$f'(s) = \frac{df}{ds} = \int_0^\infty (-t) e^{-st} F(t)\, dt = \int_0^\infty e^{-st} [-tF(t)]\, dt$$

$$\text{i.e. } f'(s) = L\{-tF(t)\} \quad \text{—— (5a)}$$

Carrying out the process of differentiation n times, we get

$$f^n(s) = \frac{d^n}{ds^n} f(s) = L\{(-1)^n t^n F(t)\} = (-1)^n L\{t^n F(t)\}$$

$$\text{or } (-1)^n f^n(s) = L\{t^n F(t)\} \quad \text{—(5b)}$$

<u>Laplace Transform of the Derivative of a function</u>

let $F(t)$ be a continuous differentiable function and $\frac{dF(t)}{dt}$ is its first derivative. If $F(t)$ and $\frac{dF(t)}{dt}$ are Laplace transformable, then the Laplace transform of the derivative $\frac{dF(t)}{dt}$ is given by

$$L\left\{\frac{dF(t)}{dt}\right\} = sL\{F(t)\} - F(0) = sf(s) - F(0)$$

where $F(0)$ is the value of $F(t)$ at $t=0$, and $f(s) = L\{F(t)\}$

Proof :- $L\left\{\frac{dF(t)}{dt}\right\} = \int_0^\infty \frac{dF(t)}{dt} e^{-st}\, dt$

Integrating by parts, we get

$$L\left\{\frac{dF(t)}{dt}\right\} = \left[F(t)e^{-st}\right]_0^\infty + s\int_0^\infty e^{-st}F(t)\,dt$$

$$= -F(0) + sL\{F(t)\} = sf(s) - F(0) \quad \text{—①}$$

This theorem is specially useful in solving differential equation with constant coefficients.

Extension of theorem :- If $F(t)$ is continuous and has derivatives of order $1,2,\dots n$ which are Laplace transformable,

$$L\left\{\frac{d^nF(t)}{dt^n}\right\} = s^n L\{F(t)\} - \sum_{k=0}^{n-1} F^k(0)\, s^{n-k-1}$$

where $F^k(0) = \left[\frac{d^kF(t)}{dt^k}\right]_{t=0}$

$$\boxed{\begin{aligned} L\left\{\frac{d^2F}{dt^2}\right\} &= \frac{dF}{dt}e^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st}\frac{dF}{dt}dt \\ &= F'(0) + s(sf(s) - F(0)) \\ &= F'(0) + s^2f(s) - sF(0) \end{aligned}}$$

Proof :- Replacing $F(t)$ by $\frac{d^{n-1}F}{dt^{n-1}}$ in ①, we get

$$L\left\{\frac{d^nF(t)}{dt^n}\right\} = sL\left\{\frac{d^{n-1}F(t)}{dt^{n-1}}\right\} - \left[\frac{d^{n-1}F(t)}{dt^{n-1}}\right]_{t=0} \quad \text{—②}$$

Similarly

$$L\left\{\frac{d^{n-1}F(t)}{dt^{n-1}}\right\} = sL\left\{\frac{d^{n-2}F(t)}{dt^{n-2}}\right\} - \left[\frac{d^{n-2}F(t)}{dt^{n-2}}\right]_{t=0} \quad \text{—③}$$

Using ③ ; eqn ② becomes

$$L\left\{\frac{d^nF(t)}{dt^n}\right\} = s^2 L\left\{\frac{d^{n-2}F(t)}{dt^{n-2}}\right\} - s\left[\frac{d^{n-2}F(t)}{dt^{n-2}}\right]_{t=0} \left[\frac{d^{n-1}F(t)}{dt^{n-1}}\right]_{t=0}$$

$$= s^3 L\left\{\frac{d^{n-3}F(t)}{dt^{n-3}}\right\} - s^2\left[\frac{d^{n-3}F(t)}{dt^{n-3}}\right]_{t=0} - s\left[\frac{d^{n-2}F(t)}{dt^{n-2}}\right]_{t=0} \left[\frac{d^{n-1}F(t)}{dt^{n-1}}\right]_{t=0}$$

$$= \dots$$

$$= s^n L\{F(t)\} - s^{n-1}F(0) - s^{n-2}F'(0) \dots - sF^{n-2}(0) - F^{n-1}(0)$$

$$= s^n L\{F(t)\} - \sum_{k=0}^{n-1} F^k(0)\, s^{n-k-1} \quad \text{—④}$$

where the exponents of F are the order of derivatives of $F(t)$ w.r.t t. For Laplace transform of second derivative $n=2$, we have from ④

$$L\left\{\frac{d^2F(t)}{dt^2}\right\} = L\{F''(t)\} = s^2L\{F(t)\} - sF(0) - \left[\frac{dF}{dt}\right]_{t=0}$$

$$= s^2 L\{F(t)\} - sF(0) - F'(0) \quad \text{—⑤}$$

Cor I <u>Initial Value Theorem:-</u>

If $L\{F(t)\} = f(s)$

then $\lim_{t \to 0} F(t) = \lim_{s \to \infty} s f(s)$; provided the limit exist

Proof:- We have $L\{\frac{dF}{dt}\} = L\{F'(t)\} = sf(s) - F(0)$

or $\int_0^\infty e^{-st}\left(\frac{dF}{dt}\right)dt = sf(s) - F(0)$

taking the limit $s \to \infty$; we get

$$\lim_{s \to \infty} \int_0^\infty e^{-st}\left(\frac{dF}{dt}\right)dt = \lim_{s \to \infty}[sf(s) - F(0)]$$

If $\frac{dF}{dt}$ is continuous; then L.H.S in the limit $s \to \infty$, approaches zero. Hence

$$\lim_{s \to \infty}[sf(s) - F(0)] = 0$$

or $\lim_{s \to \infty}[sf(s)] = F(0) = \lim_{t \to 0} F(t)$ — (6)

Cor II :- <u>Final Value Theorem</u> If $L\{F(t)\} = f(s)$, then

$\lim_{t \to \infty} F(t) = \lim_{s \to 0} s f(s)$ provided the limits exist.

Proof:- We have $L\{\frac{dF}{dt}\} = \int_0^\infty e^{-st}\left(\frac{dF}{dt}\right)dt = sf(s) - F(0)$

Now $\lim_{s \to 0} \int_0^\infty e^{-st}\left(\frac{dF}{dt}\right)dt = \lim_{s \to 0}[sf(s) - F(0)]$

If $s \to 0$, then L.H.S has a limiting value of

$$\int_0^\infty \left(\frac{dF}{dt}\right)dt = [F(t)]_0^\infty = \lim_{t \to \infty} F(t) - F(0)$$

Thus $\lim_{t \to \infty} F(t) - F(0) = \lim_{s \to 0} sf(s) - F(0)$

or $\lim_{t \to \infty} F(t) = \lim_{s \to 0}[sf(s)]$. — (7)

<u>Laplace Transform of Integral:-</u>

If $f(s) = L\{F(t)\}$

Then $L\left[\int_0^t F(t)\,dt\right] = \frac{f(s)}{s}$ — (1)

Given $f(s) = L\{F(t)\}$,

let $G(t) = \int_0^t F(t)\,dt$ ———

Then $G(0) = \int_0^0 F(t)\,dt = 0$

$$G'(t) = \frac{d}{dt}\int_0^t F(t)\,dt = F(t) \quad \Rightarrow L[G'(t)] = L[F(t)]$$

By the formula of Laplace transform of derivative

$$L\{G'(t)\} = s\,L\{G(t)\} - G(0) \qquad \text{using } L[F'(t)] = s\,L[F(t)] - F(0)$$

ie $L\{F(t)\} = s\,L\{G(t)\} - 0 = s\,L\{G(t)\}$

$\therefore$ ie $\frac{1}{s}L\{F(t)\} = L\{G(t)\}$

Hence we get

$$L\left\{\int_0^t F(t)\,dt\right\} = \frac{f(s)}{s}$$

Cor I :- Multiplication by Powers of t :- If $f(s) = L\{F(t)\}$,

then $L\{t\,F(t)\} = -\frac{d}{ds}\left[L\{F(t)\}\right] = -\frac{df(s)}{ds}$ —(2a)

And in general

$$L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s) \quad \text{— (2b)}$$

Proof :- We have $f(s) = \int_0^\infty F(t)\,e^{-st}\,dt$

Differentiating both sides w.r.t s and applying Leibniz rule of differentiation under the integral sign, we get

$$\frac{df(s)}{ds} = \int_0^\infty \frac{\partial}{\partial s}\left[F(t)\,e^{-st}\right]dt = \int_0^\infty -t\,F(t)\,e^{-st}\,dt$$

Then gives $L\{t\,F(t)\} = -\frac{df(s)}{ds} = -\frac{d}{ds}\left[L\{F(t)\}\right]$

Continuing the process n times, we get

$$L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s); \quad n = 1, 2, 3 \ldots$$

Cor II Division by t. If $f(s) = L\{F(t)\}$, then

$$L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(x)\,dx$$

provided the integral exists.

Proof Let $f(s) = L\{F(t)\}$, then

$$f(s) = \int e^{-st} F(t)\,dt$$