Orthogonal Projections:
In this section we will investigate the orthogonal projection of a vector onto a subspace of $\mathbb{R}^n$.

If x is non-zero vector in $\mathbb{R}^n$ then every vector y in $\mathbb{R}^n$ can be expressed as the sum of two component vectors $\text{proj}_x y$ and $y - \text{proj}_x y$ where $\text{proj}_x y \parallel x$ & and $y - \text{proj}_x y \perp x$

Thm Projection Theorem
Let W be a subspace of $\mathbb{R}^n$. Then every vector v in $\mathbb{R}^n$ can be written uniquely in the form

$$v = w_1 + w_2 \quad - (1)$$

where $w_1 \in W$ & $w_2 \in W^\perp$
If $\{u_1, u_2, \ldots u_k\}$ is any orthonormal basis for W then

$$w_1 = (v \cdot u_1)u_1 + (v \cdot u_2)u_2 + \cdots + (v \cdot u_k)u_k \quad -(2)$$

and $w_2 = v - w_1$
Then vector w_1 in (1) is called the orthogonal projection of v onto W and is written as $\text{proj}_W v$.

Defⁿ Orthogonal Projection onto a Subspace.

Let W be a subspace of $\mathbb{R}^n$ with orthonormal basis $\{u_1, u_2 \ldots u_k\}$ and let $v \in \mathbb{R}^n$. Then the orthogonal projection of v onto W is defined to be the vector

$$\text{Proj}_W v = (v \cdot u_1) u_1 + (v \cdot u_2) u_2 \cdots + (v \cdot u_k) u_k$$

If W is the trivial subspace of $\mathbb{R}^n$. Then $\text{proj}_W v = 0$

\# Projection theorem: Let W be a subspace of $\mathbb{R}^n$. then every vector v in $\mathbb{R}^n$ can be expressed in a unique way as $w_1 + w_2$ where $w_1 = \text{proj}_W v \in W$ and $w_2 = v - \text{proj}_W v \in W^\perp$

Qn Find the orthogonal projection of $v = (-1, 4, 3)$ onto the subspace W of $\mathbb{R}^3$ spanned by the orthogonal vectors $v_1 = (1,1,0)$ & $v_2 = (-1,1,0)$

Soln For finding orthogonal projection of v onto subspace W spanned by vector $= \{(1,1,0), (-1,1,0)\}$

Firstly we find orthonormal bases

$\because \{(1,1,0), (-1,1,0)\}$ L-I Set

$u_1 = \frac{v_1}{\|v_1\|}$ $\quad u_2 = \frac{v_2}{\|v_2\|}$

orthonormal bases are $u_1 = \frac{1}{\sqrt{2}}(1,1,0)$ & $u_2 = \frac{1}{\sqrt{2}}(-1,1,0)$

Now By defn of $Proj_W v$ we have

$Proj_W v = (v \cdot u_1) \cdot u_1 + (v \cdot u_2) u_2$

$= \left\{(-1,4,3) \cdot \frac{1}{\sqrt{2}}(1,1,0)\right\} \cdot \left\{\left(\frac{1}{\sqrt{2}}(1,1,0)\right)\right\}$

$+ \left\{\left((-1,4,3) \cdot \frac{1}{\sqrt{2}}(-1,1,0)\right)\right\} \cdot \left(\frac{1}{\sqrt{2}}(-1,1,0)\right)$

$= \frac{-1+4}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}(1,1,0) + \frac{1+4}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}(-1,1,0)$

$= \frac{3}{2}(1,1,0) + \frac{5}{2}(-1,1,0) = (-1,4,0)$

Qn Let $W = \underbrace{span\{(1,-2,-1),(3,-1,0)\}}_{\to \text{ subspace of } \mathbb{R}^3}$ of $\mathbb{R}^3$. Let $v = (-1,3,2)$ find $Proj_W v$ and decompose v into $w_1 + w_2$ where $w_1 \in W$ and $w_2 \in W^\perp$. Is decomposition unique? [DU CSF-2-2018]

Soln $W = span\{(1,-2,-1),(3,-1,0)\}$

Let $v_1 = (1,-2,-1)$, $v_2 = (3,-1,0)$

$\because$ v_1 & v_2 are L.I vectors $\Rightarrow \{v_1 v_2\}$ form bases for W

firstly find By using Gram-Schmidt process orthogonal bases on the bases $\{v_1 v_2\}$

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now $u_1 = v_1 = (1,-2,-1)$

$u_2 = v_2 - \left(\frac{v_2 \cdot u_1}{\|u_1\|^2}\right) \cdot u_1 = \cancel{(3,-1,1)} -$

$= [3,-1,0] - \frac{[3,-1,0]\cdot[1,-2,-1]}{6}[1,-2,-1]$

$= (3,-1,0) - \frac{5}{6}(+1,-2,-1) = \left(\frac{13}{6}, \frac{4}{6}, \frac{5}{6}\right)$

$\Rightarrow u_2 = \left(\frac{13}{6}, \frac{4}{6}, \frac{5}{6}\right)$ & $u_1 = (1,-2,-1)$

so u_1, u_2 are orthogonal ~~vectors~~ now find orthonormal vector

$k_1 = \frac{u_1}{\|u_1\|}$, $k_2 = \frac{u_2}{\|u_2\|}$ k_1, k_2 are orthonormal vector

$\Rightarrow k_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{6}}(1,-2,-1)$ $k_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{210}}(13,4,5)$

$\Rightarrow \left\{\frac{1}{\sqrt{6}}(1,-2,-1), \frac{1}{\sqrt{210}}(13,4,5)\right\}$ is an orthonormal basis for W

$\Rightarrow$ By defn

$w_1 = \text{proj}_W v = \cancel{(v \cdot u)}\ (v\cdot k_1)\cdot k_1 + (v\cdot k_2)\cdot k_2$

$= \left\{(-1,3,2)\cdot\frac{1}{\sqrt{6}}(1,-2,-1)\right\}\left(\frac{1}{\sqrt{6}}(+1,-2,-1)\right]$

$+ \left\{(-1,3,2)\cdot\left(\frac{13}{\sqrt{210}}, \frac{4}{\sqrt{210}}, \frac{5}{\sqrt{210}}\right)\right\}\cdot\frac{1}{\sqrt{210}}(13,4,5)$

$= \left(\frac{-33}{35}, \frac{111}{35}, \frac{12}{7}\right)$

$w_2 = v - \text{Proj}_W v$

$= (-1,3,2) - \left[\frac{-33}{35}, \frac{111}{35}, \frac{12}{7}\right] = \left[\frac{-2}{35}, \frac{-6}{35}, \frac{2}{7}\right]$

$\because$ $w_2 \in W^\perp$ because w_2 is orthogonal to K_1 & K_2 $\Rightarrow$ We decomposed $v = (-1,3,2)$ as the sum of two vectors

$w_1 = \left(\frac{-33}{35}, \frac{111}{35}, \frac{12}{7}\right)$ and $w_2 = \left(\frac{-2}{35}, \frac{-6}{35}, \frac{2}{7}\right)$

The decomposition is unique $\because$ $W \cap W^\perp = \{0\}$

Qn Let W be the subspace of $\mathbb{R}^3$, whose vectors lie in the plane $2x+y+z=0$. Let $v = (-6,10,5)$ Find $\text{proj}_W v$ and decompose v into w_1 & w_2 where $w_1 \in W$ & $w_2 \in W^\perp$.

Soln $\because$ W be the subspace of $\mathbb{R}^3$, whose vectors lie in the plane $2x+y+z=0$

$\Rightarrow W = \{(x,y,z) \in \mathbb{R}^3 : 2x+y+z=0\}$

$= \{(x,y,z) \in \mathbb{R}^3,\ y = -2x-z\}$

$\cancel{= \{(x,y,z) : x, z \in \mathbb{R}\}}$

$\cancel{= \{(x, -2x-z, z)\}}$

$\cancel{= \{x(1,-2}$

From W we get two L.I vector $x_1 = (1,0,-2)$

If put $x=1, y=0$ in $2x+y+z$

and $x_2 = (0,1,-1)$ if $x=0, y=1$ in $2x+y+z$

$\Rightarrow \{x_1\, x_2\} = \{(0,1,-1),(1,0,-2)\}$ is L.I

By using Gram-Schmidt process on the set $\{(0,1,-1),(1,0,-2)\}$ to get ~~orthogonal~~ orthogonal bases for W

$v_1 = x_1 = (1,0,-2)$

$v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{\|v_1\|^2}\right) \cdot v_1 = \left(-\frac{2}{5}, 1, -\frac{1}{5}\right)$

$\Rightarrow \{v_1\, v_2\}$ is orthogonal bases for W

~~now~~ now find orthonormal bases for W

$u_1 = \frac{v_1}{\|v_1\|}$, $u_2 = \frac{v_2}{\|v_2\|}$ are the orthonormal vector

$u_1 = \frac{1}{\sqrt{5}}(1,0,-2)$, $u_2 = \frac{1}{\sqrt{30}}(-2,5,-1)$

Now By def^n

$w_1 = \text{Proj}_W v = (v\cdot u_1)\cdot u_1 + (v\cdot u_2)\cdot u_2$

$= \left[-7, \frac{19}{2}, \frac{9}{2}\right]$

Now $w_2 = v - w_1 = [-6,10,5] - \left[-7, \frac{19}{2}, \ldots\right.$

$w_2 = \left[1, \frac{1}{2}, \frac{1}{2}\right]$

$\therefore w_2 \in W^\perp$ because w_2 is orthogonal to x_1 & x_2 $\Rightarrow$ decomposed $v = (-6, 10, 5)$ as

$v = w_1 + w_2 = \left[-7, \frac{19}{2}, \frac{9}{2}\right] + \left[1, \frac{1}{2}, \frac{1}{2}\right]$

where $w_1 \in W$ & $w_2 \in W^\perp$

→ Distance from a point to a subspace

Defn Minimum distance

Let W be a subspace of $\mathbb{R}^n$ and assume all vectors in W have initial point at the origin. Let P be any point in n-dim space. Then the minimum distance from P to W is defined to be the shortest distance b/w P and the terminal pt of any vector in W.

~~Pattr~~ Minimum distance P to W is given by $= \| v - \text{proj}_W v \|$

Qn $W = \{(x, y, z) : 2x + y + z = 0\}$ find the minimum distance from the point $P(-6, 10, 5)$ to W.

Sln $P(-6, 10, 5)$ from last Qn

~~Projw~~ $\text{proj}_W v = \left[-7, \frac{19}{2}, \frac{9}{2}\right]$

$\Rightarrow$ minimum distance from $P(-6,10,5)$ to W is

$$\|v - \text{proj}_W v\| = \|(-6,10,5) - (-7,\tfrac{19}{2},\tfrac{9}{2})\|$$

$$= \|(1,\tfrac{1}{2},\tfrac{1}{2})\| = \sqrt{1^2 + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^2}$$

$$= \sqrt{\tfrac{3}{2}}$$

Qn Find $\text{proj}_W v$ & decompose v into $w_1 + w_2$ where $w_1 \in W$ & $w_2 \in W^\perp$

(a) $W = \{(x,y,z) : 3x - y + z = 0\}$ & $v = (2,1,-1)$
[Do yourself]

(b) $W = \{(x,y,z) : 3x - y + z = 0\}$ & $v = (2,2,-3)$
[DO WF-2 -2019]
[Do yourself]

Qn Find the minimum distance from the pt $P(-1,3,2)$ to the subspace
$W = \text{span}\{(1,-2,-1), (-3,1,0)\}$ in $\mathbb{R}^3$
[Do yourself]

Qn Find the min distance from the pt $P(1,4,-2)$ to the subspace
$W = \text{span}\{(x,y,z) : 2x - 5y + z = 0\}$
[Do yourself]

{Least-Squares solutions for inconsistent systems.}

\# If the system of linear equation $Ax=b$ inconsistent i.e no soln. In this section we find an approximate soln to the system $Ax=b$ if it is inconsistent.

Defn Least-Square soln

Let $Ax=b$ be system of linear eqn where $A_{m\times n}$ matrix and $b \in \mathbb{R}^m$. A vector $v \in \mathbb{R}^n$ is said to be a least-squares soln to the system $Ax=b$ if the following condition is satisfied

$$\|Av-b\| \le \|Az-b\| \quad \forall\, z \in \mathbb{R}^n$$

In other words, v is a least-square soln to the system $Ax=b$ if Av is the ~~closed~~ closest vector in $\mathbb{R}^m$ to b.

Thm Let $Ax=b$ is a system of linear eqn $A_{m\times n}$ matrix, and $b \in \mathbb{R}^m$. Let W be the subspace of $\mathbb{R}^m$ given by $W=\{Ax : x \in \mathbb{R}^n\}$. Let $v \in \mathbb{R}^n$ then the three conditions are equivalent:

(1) v is a least-square soln to the system $Ax=b$

(2) v satisfies $(A^TA)\cdot v = A^Tb$

(3) v satisfies $Av = \text{Proj}_W b$

* Least square sol^n to the system $Ax=b$ can be found by solving the linear system
$$(A^T \cdot A)\cdot x = A^T b$$

Qn find a least-square sol^n for the linear system $Ax=b$ where

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix}$$ (DU GE-2, 2017)

Sol^n find a least-square sol^n for the system $Ax=b$, we need to solve the linear system $(A^TA)x = A^Tb$

Now $$A^TA = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 21 & 9 \\ 9 & 11 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 26 \\ 19 \end{bmatrix}$$

$$\Rightarrow A^Tb = \begin{bmatrix} 26 \\ 19 \end{bmatrix}$$

Now the system is $(A^TA)x = A^Tb$

$\Rightarrow \begin{bmatrix} 21 & 9 \\ 9 & 11 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 26 \\ 19 \end{bmatrix}$

Augmented matrix for above system is

$$\left[\begin{array}{cc:c} 21 & 9 & 26 \\ 9 & 11 & 19 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{21}R_1} \left[\begin{array}{cc:c} 1 & 3/7 & 26/21 \\ 9 & 11 & 19 \end{array}\right]$$

$R_2 \to R_2 - 9R_1$

$$\left[\begin{array}{cc:c} 1 & 3/7 & 26/21 \\ 0 & 50/7 & 55/7 \end{array}\right] \xrightarrow{R_2 \to \frac{7}{50}R_2} \left[\begin{array}{cc:c} 1 & 3/7 & 26/21 \\ 0 & 1 & 11/10 \end{array}\right]$$

$R_1 \to R_1 - \frac{3}{7}R_2$

$$\left[\begin{array}{cc:c} 1 & 0 & 23/30 \\ 0 & 1 & 11/10 \end{array}\right] \sim \left[\begin{array}{cc:c} 21 & 9 & 26 \\ 9 & 11 & 19 \end{array}\right]$$

$\Rightarrow x = \begin{bmatrix} 23/30 \\ 11/10 \end{bmatrix} = \begin{bmatrix} 0.77 \\ 1.1 \end{bmatrix}$ is the required least square soln

Qn Prove that the least-square soln for the linear system $Ax = b$ when

$A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 4 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$

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satisfies $\|A\mathbf{v}-b\| \le \|Az-b\|$

where $z = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

[DU. GE-2 2016, 2019]

<u>soln</u> To find a least square solⁿ we need to solve the linear system $(A^T A)\mathbf{v} = A^T b$

Now $A^T A = \begin{bmatrix} 2 & 1 & 4 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 21 & 9 \\ 9 & 11 \end{bmatrix}$

$\Rightarrow A^T A = \begin{bmatrix} 21 & 9 \\ 9 & 11 \end{bmatrix}$

$A^T b = \begin{bmatrix} 2 & 1 & 4 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 26 \\ 19 \end{bmatrix}$

$\Rightarrow (A^T A)\mathbf{v} = A^T b$

$\Rightarrow \begin{bmatrix} 21 & 9 \\ 9 & 11 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 26 \\ 19 \end{bmatrix}$

Augmented matrix for this system is

$\left[\begin{array}{cc:c} 21 & 9 & 26 \\ 9 & 11 & 19 \end{array}\right] \sim \left[\begin{array}{cc:c} 1 & 0 & 23/30 \\ 0 & 1 & 11/10 \end{array}\right]$

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$\Rightarrow V = \begin{bmatrix} 23/10 \\ 11/10 \end{bmatrix}$ is the desired least-squares soln.

now

$\|AV - b\| \leq \|Az - b\|$ where $z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\cancel{AV = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}}$

$\cancel{AV - b = \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}} -$

$$AV = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 23/30 \\ 11/10 \end{bmatrix} = \begin{bmatrix} \frac{46}{30} + \frac{33}{10} \\ \frac{23}{30} - \frac{11}{10} \\ \frac{92}{30} + \frac{11}{10} \end{bmatrix}$$

$$AV - b = \begin{bmatrix} \frac{46+99}{30} \\ \frac{23-33}{30} \\ \frac{92+33}{30} \end{bmatrix} - \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -1/6 \\ -1/3 \\ 1/6 \end{bmatrix}$$

$$AV - b = \begin{bmatrix} -1/6 \\ -1/3 \\ 1/6 \end{bmatrix} \qquad \|AV - b\| = \frac{\sqrt{6}}{6}$$

now $\quad Az - b = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$

$$= \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix} - \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow Az - b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \|Az - b\| = 1$$

Clearly $\frac{\sqrt{6}}{6} < 1$

$$\Rightarrow \|Au - b\| < \|Az - b\| \quad [H.P]$$

Qn find a least-squares solⁿ to the linear system $Ax = b$ where

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 1 & 3 \\ 2 & -7 & 9 \end{bmatrix} \text{ and } b = \begin{bmatrix} 9 \\ 8 \\ -1 \end{bmatrix}$$

Solⁿ To find a least-square solⁿ to $Ax = b$ we need to solve the system

$$(A^T A) \cdot x = A^T \cdot b$$

$$A^T A = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 1 & -7 \\ -1 & 3 & 9 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ 4 & 1 & 3 \\ 2 & -7 & 9 \end{bmatrix} = \begin{bmatrix} 24 & -4 & 28 \\ -4 & 59 & -63 \\ 28 & -63 & 91 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 1 & -7 \\ -1 & 3 & 9 \end{bmatrix} \begin{bmatrix} 9 \\ 8 \\ -1 \end{bmatrix} = \begin{bmatrix} 48 \\ 42 \\ 6 \end{bmatrix}$$

$$\Rightarrow A^T b = \begin{bmatrix} 48 \\ 42 \\ 6 \end{bmatrix}$$

Now $(A^T A) \cdot x = A^T b$

$$\begin{bmatrix} 24 & -4 & 28 \\ -4 & 59 & -63 \\ 28 & -63 & 91 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 48 \\ 42 \\ 6 \end{bmatrix}$$

Augmented matrix is

$$\left[\begin{array}{ccc|c} 24 & -4 & 28 & 48 \\ -4 & 59 & -63 & 42 \\ 28 & -63 & 91 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 15/7 \\ 0 & 1 & -1 & 6/7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15/7 \\ 6/7 \\ 0 \end{bmatrix}$$

$x_1 + x_3 = 15/7 \qquad x_2 - x_3 = 6/7$

$x_3 = c$ (arbitrary)

Solⁿ set $= \{ (x_1, x_2, x_3) : x_1 + x_3 = 15/7,\ x_2 - x_3 = 6/7,\ x_3 = c \}$

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$= \left\{ \left(\frac{15}{7} - x_3, \frac{6}{7} + x_3, x_3 \right) : x_3 = c \in R \right\}$

$= \left\{ \left(\frac{15}{7} - c, \frac{6}{7} + c, c \right) : c \in R \right\}$

$\Rightarrow$ This system has infinite soln

Qn find a least square soln to the inconsistent system $Ax = b$ where

$$A = \begin{bmatrix} 5 & 2 \\ 3 & 1 \\ 4 & 3 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 12 \\ 15 \\ 4 \end{bmatrix}$$

[DU. GE-2 2018]

[Do your self]

Qn $A = \begin{bmatrix} 1 & -1 \\ 4 & 1 \\ 2 & 3 \end{bmatrix}$ $b = \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}$ and $z = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

find a vector x satisfying the inequality $\|Ax - b\| \le \|Az - b\|$

[DU GE-2 2019
Do your self]

{ Syllabus complete }

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