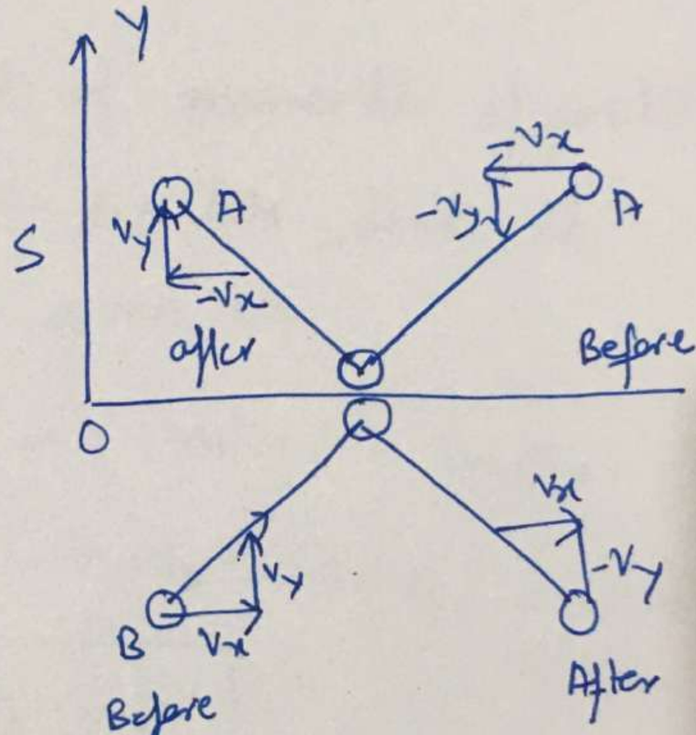
Conversion of Momentum at Relativistic Speeds:



Collision at S frame.

We assume that the ~~vel~~ law of momentum conservation is valid even at relativistic speeds.

Consider the collision of two particles A & B of equal masses m_0 in X-Y plane. Let the two particles initially have equal & opposite velocity in a frame S.

Now, if they collide elastically, their x - components of velocity will not change due to collision but y - components will have opposite sign.

The changes of momenta of particles A and B along Y - direction due to collision will be given by,

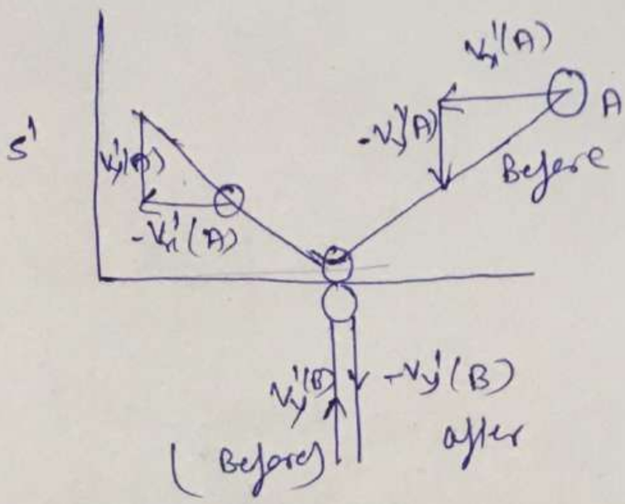
$$\Delta p_a = m_0 v_y - (-m_0 v_y) = 2 m_0 v_y$$

$$\Delta p_b = -m_0 v_y - (m_0 v_y) = -2 m_0 v_y$$

$$\Rightarrow \boxed{\Delta p_a + \Delta p_b = 0}$$

Thus Newtonian momentum is consered along Y axis along with in X - axis.

Now, we see the collision of the same particles from a frame S', which is moving with velocity v_x along X - axis relative to S. Now, using Lorentz Transformation ;



For particle A, having $-v_x$ and $-v_y$ velocity components in S frame

$$v_x'(A) = \frac{-v_x - v_x}{1 + v_x^2/c^2} = \frac{-2v_x}{1+v_x^2/c^2}$$

$$\left(\text{as } v_x' = \frac{v_x - v}{1 - v v_x/c^2}\right)$$

$$v_y'(A) = \frac{-v_y}{\gamma\left(1+v_x^2/c^2\right)}$$

(2) For particle B,

$$v_x'(B) = \frac{v_x - v_x}{1 - v_x^2/c^2} = 0\ , \quad v_y'(B) = \frac{v_y}{\gamma\left(1 - v_x^2/c^2\right)}$$

By the collision the x-components of the velocities of particle in frame s' are not changed so that the change in momentum of the system along X-axis is zero. But after collision the signs of $v_y'(A)$ and $v_y'(B)$ are change, so momenta of particles along Y-axis will be given by;

$$\Delta p_a' = \frac{2m_0 v_y}{\gamma\left(1+\frac{v_x^2}{c^2}\right)} \qquad \Delta p_b' = \frac{-2m_0 v_y}{\gamma\left(1-\frac{v_x^2}{c^2}\right)}$$

$$\Rightarrow \boxed{\Delta p_a' + \Delta p_b' \neq 0}$$

Thus in frame s', there occurs a charge in total momentum of the system. Hence, if masses of the particles are assumed to be const. at relativistic speed, Then law of conservation of momentum is not valid in all inertial frames.

So, for validity, one has assume that the masses of a particle depends on velocity relative to frame of reference. Now, if in frame s' the masses of particles A and B have m_1 & m_2 then

$$\Delta p_a' + \Delta p_b' = \frac{2m_1 v_y}{\gamma\left(1+\frac{v_x^2}{c^2}\right)} - \frac{2m_2 v_y}{\gamma\left(1-\frac{v_x^2}{c^2}\right)} = 0$$

$$\Rightarrow \frac{m_1}{m_2} = \frac{1+\frac{v_x^2}{c^2}}{1-\frac{v_x^2}{c^2}}$$

$$\Rightarrow m_1 = \frac{m_2}{\sqrt{1-\frac{v_x'^2}{c^2}}}$$

If. B is in rest in frame s', so $m_2 = m_0$ and $v_x' = v$

$$\Rightarrow \boxed{m_1 = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}}$$

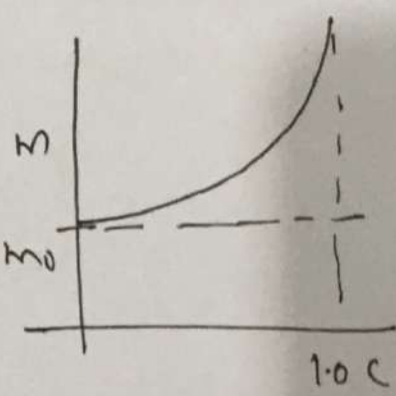
when $v=0$, $m_1 = m_0$. this is called rest mass or proper mass.

$$\Rightarrow \boxed{m = \frac{m_0}{\sqrt{1-v^2/c^2}}}$$

which represents variation of mass with velocity. So, relativistic momentum,

$$P = mv = \frac{m_0 v}{\sqrt{1-v^2/c^2}}$$

For $v/c \ll 1 \Rightarrow \boxed{P = m_0 v}$



Variation of mass with velocity

Relativistic Energy: Mass energy relation ($E = mc^2$)

Suppose a force $F = \frac{d}{dt}(mv)$ be acting on a particle of mass m so that the K.E increases. The gain in energy will be equal to work done on particles. If force displaces the particle through a distance dr along its line of action. Then

$$dE_k = F.dr = \frac{d(mv)}{dt}.dr = v\,d(mv)$$

If particle starts from rest ($v=0$) and acquires velocity v under the action of force, then

$$E_k = \int dE_k = \int_0^{u} v\, d(mv)$$

$$= vmv\Big|_0^{u} - \int_0^{v} mv\,dv = mv^2 - \underline{\underline{\int_0^{u} \frac{m_0 v\,dv}{\sqrt{1-\frac{v^2}{c^2}}}}}$$

$$= \frac{m_0 u^2}{\sqrt{1-\frac{u^2}{c^2}}} + m_0 c^2 \sqrt{1-\frac{u^2}{c^2}} - m_0 c^2$$

(Note: Solving underline integration, we can get)

$$\boxed{E_k = (m-m_0)c^2}$$

where $m = \dfrac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$

This is the expression for relativistic K.E.

⇒ Total Energy

$$E = K.E + \text{Rest Energy}$$

$$= (m-m_0)c^2 + m_0 c^2$$

$$\boxed{E = mc^2 = \frac{m_0 c^2}{\sqrt{1-\frac{u^2}{c^2}}}}$$

This is called relativistic energy of particle, having relativistic mass m. This is called <u>Einstein's mass energy relation</u>.

Relation B/w Momentum and energy / conservation laws

we have

$$p = \frac{m_0 v}{\sqrt{1-v^2/c^2}} \Rightarrow p^2 = m_0^2 v^2 \gamma^2$$

$$\gamma^2 = \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 \Rightarrow \gamma^2 v^2 = \gamma^2 c^2 - c^2$$

so $\quad p^2 = m_0^2(\gamma^2 c^2 - c^2)$

$$= m_0^2 \gamma^2 c^2 - m_0^2 c^2$$

$$\Rightarrow p^2 c^2 = m_0^2 \gamma^2 c^4 - m_0^2 c^4$$

$$p^2 c^2 = E^2 - m_0^2 c^4$$

$$E = mc^2 = m_0 \gamma c^2 \Rightarrow E^2 = m_0^2 \gamma^2 c^4$$

$$\Rightarrow \boxed{E^2 = p^2 c^2 + m_0^2 c^4}$$ → This is const. quantity.