Lengths of plane Curve

Length of a parametrically defined curve –

Let a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \le t \le b$ where f' and g' are Continuous and not simultaneously zero on $[a, b]$, such a curve is called Smooth Curve. If the curve is traversed Exactly once as t increases from $t = a$ to $t = b$ then the arc length of C between $(f(a), g(a))$ and $(f(b), g(b))$ is given by –

$$L = \int_a^b \sqrt{\{f'(t)\}^2 + \{g'(t)\}^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

e.g. find the length of the curve

$x = r\cos t$ and $y = r\sin t$, $0 \le t \le 2\pi$

Solⁿ As t varies from 0 to 2π, the Circle traversed Exactly once, so the length of the circle is

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Now

$$\frac{dx}{dt} = -r\sin t \text{ and } \frac{dy}{dt} = r\cos t$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = r^2(\sin^2 t + \cos^2 t) = r^2$$

$$\therefore\ L = \int_0^{2\pi} \sqrt{r^2}\, dt = \int_0^{2\pi} r\, dt = r[t]_0^{2\pi} = 2\pi r.$$

Arc Length for $y = f(x)$

Let f be a function with a continuous first derivative on $[a,b]$. then the length of the curve $y = f(x)$ from $x = a$ to $x = b$ is given by-

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b \sqrt{1+\{f'(x)\}^2}\,dx$$

e.g. Find the length of the curve $f(x) = x^{3/2}$ over the interval $[0,1]$

Soln $f(x) = x^{3/2}$, $0 \le x \le 1$

$\Rightarrow f'(x) = \frac{3}{2}x^{1/2}$, which is continuous on $[0,1]$

$$L = \int_0^1 \sqrt{1+\{f'(x)\}^2}\,dx = \int_0^1 \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\,dx$$

$$= \frac{1}{27}\left[(13)^{3/2} - 8\right].$$

Formula for the Arc length of $x = g(y)$, $c \le y \le d$

Let g be a function with continuous first derivative on ~~[a, b]~~ $[c,d]$ then the length of the curve $x = g(y)$ from $y = c$ to $y = d$ is given by-

$$L = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = \int_c^d \sqrt{1+\{g'(y)\}^2}\,dy$$

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Areas of Surfaces of Revolution -

1- Revolution about the X-Axis

Let f is a smooth and non-negative function on $[a,b]$ then the area of the surface of revolution that is generated by revolving the curve $y=f(x)$ between $x=a$ and $x=b$ about x-axis is defined by-

$$S = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b 2\pi f(x)\sqrt{1+\{f'(x)\}^2}\,dx$$

e.g. Find the area of the surface generated by revolving the curve $y=2\sqrt{x}$, $1 \le x \le 2$, about x-axis.

Soln

$$S = \int_1^2 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_1^2 2\pi\, 2\sqrt{x}\,\frac{\sqrt{x+1}}{\sqrt{x}}\,dx$$

$$= \frac{8\pi}{3}\left[3\sqrt{3}-2\sqrt{2}\right]$$

2. Revolution about y-axis -

Let g be a smooth and non-negative function on $[c,d]$ then the area of the surfance of revolution that is generated by revolving the curve $x=g(y)$ between $y=c$ and $y=d$ about the y-axis. is defined as.

$$S = \int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = \int_c^d 2\pi g(y)\sqrt{1+\{g'(y)\}^2}\,dy$$

Surface Area of Revolution for Parametric Curves

(a) Revolution about the x-axis $(y \geq 0)$

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

(b) Revolution about the y-axis $(x \geq 0)$

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

e.g. Find the area of the surface revolving the semi-circle $x = \cos t$, $y = \sin t$, $0 \leq t \leq \pi$ about x-axis.

Soln We have

$$x = \cos t, \qquad y = \sin t$$

$$\frac{dx}{dt} = -\sin t, \qquad \frac{dy}{dt} = \cos t$$

$$\Rightarrow S = \int_0^\pi 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^\pi 2\pi \sin t \sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$= 2\pi \int_0^\pi \sin t\, dt = \underline{4\pi} \text{ Ans}$$