Gradient, Divergence, Curl

Vector product :- If $u = (u_1, u_2, u_3)$ & $v = (v_1, v_2, v_3)$

Then (i) $\vec{u}\cdot\vec{v} = (u_1, u_2, u_3)\cdot(v_1, v_2, v_3)$

$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$ is called scalar or dot product.

(ii) $\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$ is called vector or cross product

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Gradient :- Let $\phi(x, y, z)$ be a scalar field

Then, the gradient is a vector field defined by

$$grad(\phi) = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = \nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k.$$

Example If $\phi(x, y, z) = x^2 + 2y + z^3$

Then
$$grad(\phi) = \nabla\phi = \frac{\partial}{\partial x}(x^2+2y+z^3)\,i + \frac{\partial}{\partial y}(x^2+2y+z^3)\,j + \frac{\partial}{\partial z}(x^2+2y+z^3)\,k$$

$$= 2xi + 2j + 3z^2k$$

$$\Rightarrow \boxed{grad(\phi) = 2xi + 2j + 3z^2k}$$

<u>Divergence</u>:- Let $\vec{F} = (f(x,y,z), g(x,y,z), h(x,y,z))$ be a vector field, continuously differentiable with respect to x, y, and z. Then the divergence of $\vec{F}$ is the scalar field defined by,

$$\boxed{\operatorname{div}(\vec{F}) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}}$$

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<u>Example</u>: If $\vec{F} = (x^2+y^2, y^2+z^2, z^2+2x)$

Then
$$\operatorname{div}(\vec{F}) = \frac{\partial}{\partial x}(x^2+y^2) + \frac{\partial}{\partial y}(y^2+z^2) + \frac{\partial}{\partial z}(z^2+2x)$$

$$= 2x + 2y + 2z$$

$$\Rightarrow \boxed{\operatorname{div}(\vec{F}) = 2x+2y+2z}$$

<u>Curl</u>:- The curl of $\vec{F} = (f(x,y,z), g(x,y,z), h(x,y,z))$ is ~~scal~~ the vector field defined by

$$\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) i + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) j + \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial y}\right) k$$

$$\Rightarrow \operatorname{curl}(\vec{F}) = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial h}{\partial x} - \frac{\partial f}{\partial y}\right).$$

Example If $\vec{F} = 2xi + y^2j + 2xzk$, Then find the curl($\vec{F}$)

Solution: $\vec{F} = 2xi + y^2j + 2xz = (2x, y^2, 2xz)$

$\Rightarrow f(x,y,z) = 2x,\ g(x,y,z) = y^2,\ h(x,y,z) = 2xz$

$\Rightarrow$ & $\frac{\partial f}{\partial x} = 2,\ \frac{\partial f}{\partial y} = 0,\ \frac{\partial f}{\partial z} = 0$

$\frac{\partial g}{\partial x} = 0,\ \frac{\partial g}{\partial y} = 2y,\ \frac{\partial g}{\partial z} = 0$

$\frac{\partial h}{\partial x} = 2z,\ \frac{\partial h}{\partial y} = 0,\ \frac{\partial h}{\partial z} = 2x$

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Now

$$\operatorname{Curl}(\vec{F}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)i + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)j + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)k$$

$$= (0-0)i + (x - 2z)j + (0-0)k$$

$$\Rightarrow \boxed{\operatorname{Curl}(\vec{F}) = -2zj}$$

Example:- If $\vec{F} = yzi + 3xzj + zk$ then find the curl$(\vec{F})$

Solution:- $\vec{F} = (yzi + 3xzj + zk) = (yz, 3xz, z)$

$\Rightarrow f(x,y,z) = yz,\ g(x,y,z) = 3xz,\ h(x,y,z) = z$

Now

$$\text{Curl}(\vec{F}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 3xz & z \end{vmatrix} = -3xi + yj + (3z - z)k$$

$$\Rightarrow \boxed{\text{Curl}(\vec{F}) = -3xi + yj + 2zk}$$

Del operator:- The vector differential operator ∇ is define as

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Example:- If $\phi(x,y,z) = x^2 + y^2 + 2xz$ then

$$\nabla\phi = \left(\frac{\partial}{\partial x} \cancel{\phi} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)\phi$$

$$= \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

$$\boxed{\nabla\phi = (2x + 2z)i + 2yj + 2xk}$$

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Property :- If $\phi(x,y,z)$ is scalar field and $\vec{F}$ is the vector field then

(i) $grad(\phi) = \nabla\phi$

(ii) $div(\vec{F}) = \nabla\cdot\vec{F}$

(iii) $curl(\vec{F}) = \nabla\times\vec{F}$

Proof (iii) Let $\vec{F} = fi + gj + zk = (f(x,y,z), g(x,y,z), h(x,y,z))$

Now $\nabla\times\vec{F} = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\times\left(fi + gj + z\hat{k}\right)$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = curl(\vec{F})$$

$\Rightarrow \boxed{curl(\vec{F}) = \nabla\times\vec{F}}$ **Proved**.

Property : If $\phi(x,y,z)$ is scalar field and $\vec{F}$ is the vector field then

(i) $div(curl(\vec{F})) = 0$ (ii) $curl(div(\phi)) = 0$

(iii) $\nabla\cdot(\nabla\times\vec{F}) = 0$ (iv) $\nabla\times(\nabla\phi) = 0$

Formula Results on Gradient:- Let ϕ, ψ be two scalar field Then

(i) $\nabla(\phi \cdot \psi) = \phi(\nabla\psi) + \psi(\nabla\phi)$

(ii) $\nabla(\phi^n) = n \cdot \phi^{n-1} \cdot \nabla\phi$

(iii) $\nabla(\phi/\psi) = \dfrac{\psi\nabla\phi - \phi\nabla\psi}{\psi^2}$