Soln: $n = 25$; $p = 0.2 = \frac{1}{5}$; $q = \frac{4}{5}$

$\mu = E(X) = np = 25 \times \frac{1}{5} = 5$

$\sigma = \sqrt{var(X)} = \sqrt{npq} = \sqrt{25 \times \frac{1}{5} \times \frac{4}{5}} = 2$

$\therefore$ ~~$P(X < \mu - 2\sigma$~~ $\mu - 2\sigma = 5 - 4 = 1$

$\therefore P(X < \mu - 2\sigma) = P(X < 1)$

$= P(X = 0)$

$= {}^{25}C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^{25}$

$= \left(\frac{4}{5}\right)^{25}$

Q. If 3% of the electric bulbs manufactured by a company are defective, Find the probability that in a sample of 100 bulbs:

(i) 2 bulbs are defective

(ii) less than or equal to 2 bulbs are defective.

Soln: $p = \frac{3}{100}$, $q = \frac{97}{100}$

(i) $P(X = 2) = {}^{100}C_2 \left(\frac{3}{100}\right)^2 \left(\frac{97}{100}\right)^{98}$

$= 4455 \times \left(\frac{97^{98}}{10^{200}}\right) = 4455 \left(\frac{97^{98}}{10^{199}}\right)$

(ii) $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$$= {}^{100}C_0 \left(\frac{3}{100}\right)^0 \left(\frac{97}{100}\right)^{100} + {}^{100}C_1 \left(\frac{3}{100}\right)^1 \left(\frac{97}{100}\right)^{99} + {}^{100}C_2 \left(\frac{3}{100}\right)^2 \left(\frac{97}{100}\right)^{98}$$

$$= \frac{97^{100}}{10^{200}} + \frac{97^{99}}{10^{198}} + 4455\left(\frac{97^{98}}{10^{199}}\right)$$

$$= \frac{97^{98}}{10^{198}} \left[\frac{97^2}{10^2} + 97 + \frac{4455}{10}\right]$$

$$= 63659\left(\frac{97^{98}}{10^{200}}\right)$$

Q. Find the probability that in five tosses of a fair die, a 3 will appear:
(i) at most once (ii) at least once.

Soln: $n = 5$; $p = \frac{1}{6}$; $q = \frac{5}{6}$

(i) $P(X \leq 1) = P(X=0) + P(X=1)$

$$= {}^{5}C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^5 + {}^{5}C_1 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^4$$

$$= \left(\frac{5}{6}\right)^5 + \left(\frac{5}{6}\right)^5$$

$$= 2\left(\frac{5}{6}\right)^5 = \frac{6250}{7776}$$

(ii) $P(X \geq 1) = p\,1 - P(X < 1)$

$= 1 - P(X = 0)$

$= 1 - \left(\frac{5}{6}\right)^5 = \frac{4651}{7776}$

Q. What is the probability of getting a total of 7
(i) at least once (ii) at most once
in three tosses of a pair of fair dice?

Soln: $n = 3$, $\cancel{p=}$

favourable cases $= \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

$\therefore\ p = \frac{6}{36} = \frac{1}{6}$, $q = \frac{5}{6}$

$\therefore$ (i) $P(X \geq 1) = P(X=1) + P(X=2) + P(X=3)$

$= 1 - P(X = 0)$

$= 1 - {}^3C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^3$

$= 1 - \frac{125}{216} = \frac{91}{216}$

(ii) $P(X \leq 1) = P(X=0) + P(X=1)$

$= \frac{91}{216} + {}^3C_1 \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2 = \frac{91}{216} + \frac{75}{216}$

$= \frac{166}{216} = \frac{83}{108}$

(45)

(2) Poisson Distribution :-

(This is also a discrete distribution)

Let X be a discrete random variable that can take on the values $0, 1, 2, \ldots$, such that the probability function of X is given by.

$$f(x) = P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!} \; ; \; x = 0, 1, 2, \ldots$$

where, λ is a given constant and is called the parameter of the Poisson distribution

This distribution is called the Poisson distribution and a random variable having this distribution is said to be Poisson distributed. This is denoted by $X \sim P(\lambda)$

Relation between the Binomial and Poisson Distributions :-

In the binomial distribution, if n is large while the probability p of occurrence of an event is close to zero, so that $q = 1-p$ is close to 1, the event is called a rare event.

In practice, we shall consider an event as rare if the number of trials is at least 50 ($n \geq 50$) while np is less than 5, and $p \leq 0.1$.

For such cases, the binomial distribution is very closely approximated by the Poisson distribution with $\lambda = np$. Therefore, we can say that Poisson distr. is a limiting case of Binomial distribution

i.e $$\lim_{\substack{n \to \infty \\ p \to 0 \\ q \to 1}} {}^nC_x p^x q^{n-x} = \frac{\lambda^x e^{-\lambda}}{x!} \quad (\lambda = np)$$

Proof:

Binomial distribution is:

$$B(n,p) = f(x) = {}^nC_x p^x q^{n-x} \; ; \; x = 0, 1, 2, \ldots, n$$

$$= {}^nC_x p^x (1-p)^{n-x}$$

$$np = \lambda \Rightarrow p = \frac{\lambda}{n}$$

$$\therefore B(n,p) = {}^nC_x \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n!}{(n-x)!\,x!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n(n-1)\ldots(n-(x-1))\cancel{(n-x)!}}{\cancel{(n-x)!}\,x!\;n^x} \cdot \lambda^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

(48)

$$= \frac{n(n-1)\cdots(n-x+1)}{n^x}\left(\frac{\lambda^x}{x!}\right)\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$\lim_{n\to\infty} B(n,p) = \left(\frac{\lambda^x}{x!}\right)\left(\lim_{n\to\infty}\frac{n(n-1)\cdots(n-x+1)}{n^x}\right).$$

$$\times\left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\right).\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \left(\frac{\lambda^x}{x!}\right)\left[\lim_{n\to\infty}\frac{\cancel{n^x}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{(x-1)}{n}\right)}{\cancel{n^x}}\right].$$

$$.\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n.\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \left(\frac{\lambda^x}{x!}\right)(1)(e^{-\lambda})(1) = \frac{\lambda^x e^{-\lambda}}{x!} = P(\lambda)$$

Because; $\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$

$$\left(\because \left(1-\frac{\lambda}{n}\right)^n = 1 - \frac{n\lambda}{n} + \frac{n(n-1)}{2!}\frac{\lambda^2}{n^2} - \frac{n(n-1)(n-2)}{3!}\frac{\lambda^3}{n^3} + \cdots\right.$$

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = 1 - \lambda + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} - \cdots = e^{-\lambda}$$

$$\left.\text{Similarly; } \left(1+\frac{\lambda}{n}\right)^n \underset{n\to\infty}{=} e^{\lambda}\right)$$

Q If X is a Poisson distributed random variable, then ~~find~~ prove that

$$E(X) = \lambda = np$$

$$Var(X) = \lambda = np$$

Soln:

$$E(X) = \sum_{x=0}^{\infty} x f(x) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\left[1\cdot\lambda + 2\cdot\frac{\lambda^2}{2!} + 3\cdot\frac{\lambda^3}{3!} + \cdots\right].$$

$$= \lambda e^{-\lambda}\left[1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right]$$

$$= \lambda e^{-\lambda}\cdot e^{\lambda} = \lambda = np$$

$$E(x^2) = \sum_{x=0}^{\infty} x^2 f(x) = \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^x}{x!} \cancel{= \sum_{x=0}^{\infty} x(x-1)+x}$$

$$= \sum_{x=0}^{\infty} (x(x-1)+x)\frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!} + E(X)$$

$$= e^{-\lambda}\left[2\cdot 1\frac{\lambda^2}{2!} + 3\cdot 2\cdot\frac{\lambda^3}{3!} + \cdots\right] + \lambda$$

$$= e^{-\lambda}\lambda^2\left[1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right] + \lambda$$

$$= e^{-\lambda}\lambda^2 e^{\lambda} + \lambda = \lambda^2 + \lambda$$

$$\therefore Var(X) = E(x^2) - (E(x))^2 = \lambda^2 + \lambda - \lambda^2$$

$$= \lambda = np$$

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Q. 10% of the tools produced in a certain manufacturing process turn out to be defective. Find the probability that in a sample of 10 tools chosen at random, exactly 2 will be defective, by using (a) the binomial distribution (b) the Poisson approximation to the binomial distribution.

Soln: (a) $p = \frac{10}{100} = 0.1$; $q = 0.9$

$n = 10$

$P(X=2) = {}^{10}C_2 (0.1)^2 (0.9)^8$

$= 0.1937 \simeq 0.19$

(b) $\lambda = np = (10)(0.1) = 1$

Poisson distrn:

$\therefore P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}$

$P(X=2) = \frac{\lambda^2 e^{-\lambda}}{2!} = \frac{(1)^2 e^{-1}}{2!} = 0.1839$
$\simeq 0.18$

Q. If the probability that an individual will suffer a bad reaction from injection of a given serum is 0.001, Determine the probability that out of 2000 individuals,

(a) exactly 3 (b) more than 2, individuals will suffer a bad reaction.

Sol^n: Let X denote the no. of individuals suffering a bad reaction.

Here, $n = 2000$, $p = 0.001$

$$\lambda = np = 2$$

∴ Bad reactions can be assumed to be rare events, we can suppose that X is Poisson distributed. i.e

$$P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{2^x e^{-2}}{x!} \;;\; x = 0,1,2,\ldots,n$$

(a) $P(X=3) = \dfrac{2^3 e^{-2}}{3!} = 0.180$

(b) $P(X>2) = 1 - [P(X=0) + P(X=1) + P(X=2)]$

$$= 1 - \left[\frac{2^0 e^{-2}}{0!} + \frac{2^1 e^{-2}}{1!} + \frac{2^2 e^{-2}}{2!}\right]$$

$$= 1 - 5e^{-2} = 0.323$$

Note: An exact evaluation of the probabilities using the binomial distribution would require much more labor.

Q. A telephone switch board handles 600 calls, on an average, during a rush hour. The board can make a maximum of 20 connections per minute. Use the Poisson distrin to evaluate the probability that the board will be overtaxed during any given minute.

Soln: $\lambda = \frac{600}{60} = 10$ per minute (i.e average)

Required probability $= P(X > 20)$

$= 1 - P(X \leq 20)$

$= 1 - \sum_{x=0}^{20} \frac{e^{-10} 10^x}{x!}$

Q. If r has a Poisson distribution such that:

$$P(r=2) = \frac{3}{2} P(r=1)$$

then find the ratio $\frac{P(r=3)}{P(r=0)}$

Soln: $P(r=2) = \frac{3}{2} P(r=1)$

$\Rightarrow \frac{\cancel{e^{-\lambda}} \lambda^2}{\cancel{2!}} = \frac{3}{2} \cdot \frac{\cancel{e^{-\lambda}} \lambda}{1!} \Rightarrow \lambda^2 - 3\lambda = 0$

$\lambda = 0, 3$; $\lambda = 0$ is not acceptable

$\therefore \lambda = 3$

$$\frac{P(r=3)}{P(r=0)} = \frac{\frac{e^{-\lambda}\lambda^3}{3!}}{\frac{e^{-\lambda}\lambda^0}{0!}} = \frac{\lambda^3}{3!} = \frac{3^3}{3!} = \frac{9}{2}$$

(3) The Normal (or Gaussian) Distribution:-

(This is a continuous distribution)

The Normal distribution is defined by the density function:

$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \; ; \; -\infty < x < \infty} \quad (1)$$

A random variable X having this distribution is said to be Normally distributed.

- Normal distribution is denoted by: $N(\mu, \sigma^2)$
- The symbols μ & σ are used to represent the parameters of the normal distribution as they turn out to be the mean and standard deviation, respectively, of the distribution.

(54)

The distribution function of Normal distribution $N(\mu, \sigma)$ is given by:

$$F(x) = P(X \leq x) = \cancel{\frac{1}{\sigma\sqrt{2\pi}}} \int_{-\infty}^{x} f(x)\,dx$$

$$\Rightarrow \boxed{F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot dx} \quad - (2)$$

→ Also $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

Standard Normal Distribution:

If we let Z be the standardized variable corresponding to X i.e if we let

$$Z = \frac{X-\mu}{\sigma}$$

then $E(Z) = 0$; $Var(Z) = 1$.

In such cases, the density function for Z can be obtained (1) by placing $\mu = 0$ & $\sigma = 1$

$$\boxed{f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}} \quad (-\infty < z < +\infty) \quad - (3)$$

This is referred to as the standard normal density function & this dis^n is denoted as $Z \sim N(0,1)$

i.e $N(\mu,\sigma) \xrightarrow[z=\frac{x-\mu}{\sigma}]{} N(0,1)$

~~Proof of (3):~~ put $z = \frac{x-\mu}{\sigma}$ in (1)

$dx = \sigma\, dz$

then $f(z) = \frac{1}{\sigma\sqrt{2\pi}} e$

The corresponding distribution function is:

$$\boxed{F(z) = P(Z \le z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{z^2}{2}} dz} \quad (4)$$

Since: $\int_{-\infty}^{+\infty} f(z) dz = 1 \Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz = 1$

$$\Rightarrow \boxed{\int_{-\infty}^{+\infty} e^{-\frac{z^2}{2}} dz = \sqrt{2\pi}} \quad (5)$$

$\Rightarrow$ Since $e^{-z^2/2}$ is an even function

$$\therefore \boxed{\int_{-\infty}^{0} e^{-\frac{z^2}{2}} dz = \int_{0}^{\infty} e^{-\frac{z^2}{2}} dz = \frac{\sqrt{2\pi}}{2} = \sqrt{\frac{\pi}{2}}} \quad (6)$$

$\therefore$ from (4): $F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{z^2}{2}} dz + \frac{1}{\sqrt{2\pi}} \int_{0}^{z} e^{-\frac{z^2}{2}} dz$

$$\boxed{F(z) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_{0}^{z} e^{-z^2/2} dz} \quad (7)$$

error function is defined by.

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-z^2} dz \quad \text{--- (8)}$$

∴ (7) can be written as;

$$\boxed{F(z) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right)\right]} \quad \text{--- (9)}$$

<u>Theorem</u>: If $X \sim N(\mu, \sigma)$ then

$$P(a \leq X \leq b) = F\left(\frac{b-\mu}{\sigma}\right) - F\left(\frac{a-\mu}{\sigma}\right)$$

<u>Proof</u>: $$P(a \leq X \leq b) = \frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

put $z = \frac{x-\mu}{\sigma} \Rightarrow dx = \sigma dz$

when $x = a \Rightarrow z_1 = \frac{a-\mu}{\sigma}$

when $x = b \Rightarrow z_2 = \frac{b-\mu}{\sigma}$

$$\therefore P[a < x < b] = \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-\frac{z^2}{2}} dz \quad \text{--- (1)}$$

distribution funⁿ:

$$F(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$