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B.Sc (H) Sec-A

Electromagnetic Potentials & Gauge Invariance

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \frac{4\pi\vec{J}}{c} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

Consider first statics.

Electrostatics

$$\vec{\nabla}\times\vec{E} = 0 \quad \text{since } \frac{\partial\vec{B}}{\partial t} = 0 \text{ for statics}$$

From vector calculus we know that if the curl of a vector is ~~also~~ everywhere zero, then we can alway write that vector field as the gradient of some scalar function ϕ

$$\vec{E} = -\vec{\nabla}\phi \Rightarrow \vec{\nabla}\times E = -\vec{\nabla}\times(\vec{\nabla}\phi) = 0$$

ϕ is electrostatic potential

Gauss Law becomes

$$\vec{\nabla}\cdot\vec{E} = -\vec{\nabla}\cdot(\vec{\nabla}\phi) = -\nabla^2\phi = 4\pi\rho$$

$$\boxed{\nabla^2\phi = -4\pi\rho} \quad \text{Poisson's Equation}$$

In regions where $\rho = 0$, we have

$$\nabla^2\phi = 0 \quad \text{Laplace's Equation}$$

In our discussion of Coulomb's Law we saw that the electric field from a distribution of localized charges was

$$\vec{E}(\vec{r}) = \int d^3r' \, \rho(\vec{r}') \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

$$= -\vec{\nabla}\left[\int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\right] = -\vec{\nabla}\phi$$

We therefore see that the solution to Poisson's equ for a localized charge distribution ρ (with $\vec{E}=0$ as $\vec{r}\to\infty$) is

$$\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

We will soon spend a fair amount of time learning new ways to solve $\nabla^2\phi=\rho$, both for arbitrary ρ where we want an approx to the above integral (multipole expansion), and for cases where ϕ or $\vec{\nabla}\phi$ are predetermined on the surfaces of specified regions of space, such as conducting surfaces (boundary value problems).

Magnetostatics

$$\vec{\nabla}\cdot\vec{B} = 0$$

From vector calculus we know that if the divergence of a vector field vanishes everywhere, then it can always be written as the curl of another vector field $\vec{A}$

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad\Rightarrow\quad \vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$$

$\vec{A}$ is the <u>magnetic vector potential</u>

This remains true in <u>general</u> - not just in magneto<u>statics</u>

Ampere's Law becomes

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} \qquad \left(\text{in magnetostatics } \frac{\partial\vec{E}}{\partial t} = 0\right)$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) = \frac{4\pi}{c}\vec{j}$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \frac{4\pi}{c}\vec{j}$$

<u>magnetostatic gauge invarience</u>

There are many possible vector potentials $\vec{A}$ that result in the same $\vec{B}$. If $\vec{A}$ is such that $\vec{\nabla}\times\vec{A} = \vec{B}$, then $\vec{A}' = \vec{A} + \vec{\nabla}\chi$ also gives $\vec{\nabla}\times\vec{A}' = \vec{B}$ since $\vec{\nabla}\times\vec{\nabla}\chi = 0$ for <u>any</u> scalar function $\chi(\vec{r})$.

Therefore we can always choose to represent $\vec{B}$ by a vector potential $\vec{A}$ such that $\vec{\nabla}\cdot\vec{A}=0$.

proof: Suppose we had $\vec{B}=\vec{\nabla}\times\vec{A}$ for some $\vec{A}$ with $\vec{\nabla}\cdot\vec{A}=D(\vec{r})\neq 0$. Construct an $\vec{A}'=\vec{A}+\vec{\nabla}\chi$ with χ chosen as follows:

$$\vec{\nabla}\cdot\vec{A}'=\vec{\nabla}\cdot\vec{A}+\nabla^2\chi=0 \Rightarrow \nabla^2\chi=-\vec{\nabla}\cdot\vec{A}=D$$

Solve for χ, for exaple

$$\chi(\vec{r})=\int d^3r'\frac{D(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|}$$

we thus have constucted an $\vec{A}'$ such that $\vec{\nabla}\times\vec{A}'=\vec{B}$ and $\vec{\nabla}\cdot\vec{A}'=0$.

This freedom to choose various $\vec{A}$'s that give the same $\vec{B}$ is called gauge invarience. Imposing a particular additional constraint on $\vec{A}$ that removes this freedom is called fixing the gauge. The choice $\vec{\nabla}\cdot\vec{A}=0$ is usually known as the Coulomb gauge (or sometimes the Landau gauge). Going from $\vec{A}$ to $\vec{A}'=\vec{A}+\vec{\nabla}\chi$ is called making a gauge transformation.

"Working in the Coulomb gauge" with $\vec{\nabla}\cdot\vec{A}=0$, Amperes Law becomes

$$\boxed{\nabla^2\vec{A}=-\frac{4\pi}{c}\vec{j}} \qquad \text{Poisson's Equ.}$$

For a localized current density

$$\vec{A}(\vec{r})=\frac{1}{c}\int d^3r'\frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

Back to dynamics

$$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \boxed{\vec{B} = \vec{\nabla}\times\vec{A}} \text{ remains true}$$

But now instead of $\vec{\nabla}\times\vec{E} = 0$ we have

$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$

$$\Rightarrow \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) = 0$$

$$\Rightarrow \vec{\nabla}\times\left(\vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) = 0$$

$\Rightarrow$ there exists a scalar potential ϕ such that

$$\vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\phi \quad \text{or} \quad \boxed{\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}}$$

Gauss's law for electric field now becomes

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho = -\nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = 4\pi\rho$$

$$\boxed{\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -4\pi\rho}$$ Gauss law in terms of electromagnetic potentials

Ampere's law becomes

$$\vec{\nabla}\times\vec{B} = \frac{4\pi\vec{J}}{c} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \frac{4\pi\vec{J}}{c} - \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\phi - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}$$

$$\boxed{-\nabla^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = \frac{4\pi}{c}\vec{j} - \frac{1}{c}\frac{\partial}{\partial t}\left(\vec{\nabla}\phi + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right)}$$

or
$$\boxed{-\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{j} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right)}$$

gauge invarience

As before, we can always construct $\vec{A}' = \vec{A} + \vec{\nabla}\chi$, for any scalor function χ, that gives the same $\vec{B}$. But since $\vec{A}$ now also enters expression for $\vec{E}$, we need to make sure that if we change $\vec{A}$ to $\vec{A}'$, we must make some corresponding change ϕ to ϕ' so that $\vec{E}$ does not change.

$$\left[\begin{aligned} \vec{A}' &= \vec{A} + \vec{\nabla}\chi \\ \phi' &= \phi - \frac{1}{c}\frac{\partial \chi}{\partial t} \end{aligned}\right] \quad \text{gauge transformation}$$

For any scalor χ, the above $\vec{A}'$ and ϕ' give the same values of $\vec{E}$ and $\vec{B}$ as $\vec{A}$ and ϕ.

Proof:
$$\vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{\nabla}\chi = \vec{\nabla}\times\vec{A} = \vec{B}$$

$$\left(-\vec{\nabla}\phi' - \frac{1}{c}\frac{\partial\vec{A}'}{\partial t}\right) = -\vec{\nabla}\phi + \frac{1}{c}\vec{\nabla}\frac{\partial\chi}{\partial t} - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\chi$$

$$= \left(-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) = \vec{E}$$

As before, we can fix the gauge by imposing some additional constraint on $\vec{A}$ and ϕ. There are two popular choices:

1) <u>Lorentz Gauge</u>

<u>gauge constraint</u>: require $\frac{1}{c}\frac{\partial \phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0$

Then Gauss' Law becomes

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -4\pi\rho$$

$$\Rightarrow \nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right) = -4\pi\rho$$

$$\boxed{\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho}$$

Ampere's Law becomes

$$-\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{J} - \vec{\nabla}\underbrace{\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right)}_{=0}$$

$$\boxed{\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{J}}$$

The combination $-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \equiv \Box^2$ is the <u>wave equation</u> operator.

In Lorentz gauge, $\vec{A}$ and ϕ satisfy the inhomogeneous wave equations:

$$\boxed{\begin{aligned}\Box^2\vec{A} &= \frac{4\pi}{c}\vec{J}\\ \Box^2\phi &= 4\pi\rho\end{aligned}}$$

when $\vec{J}=0, \rho=0$ electromagnetic waves are solution!

Note: Lorentz gauge condition does not uniquely determine $\vec{A}$ and ϕ. If one ~~constructs~~ has $\vec{A}$ and ϕ obeying Lorentz gauge condition, and then constructs

$$\vec{A}' = \vec{A} + \vec{\nabla}\chi$$
$$\phi' = \phi - \frac{1}{c}\frac{\partial\chi}{\partial t}$$

Then $\vec{A}'$ and ϕ' will also be in Lorentz gauge provided $\Box^2\chi = 0$ (proof left to reader)

2) Coulomb Gauge

gauge constraint: require $\vec{\nabla}\cdot\vec{A} = 0$
if $\vec{A}$ is in the Coulomb Gauge, then $\vec{A}' = \vec{A} + \vec{\nabla}\chi$ will also be in Coulomb gauge provided $\nabla^2\chi = 0$.

Then Gauss' law becomes

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -4\pi\rho$$

$$\Rightarrow \boxed{\nabla^2\phi = -4\pi\rho} \quad \text{same as electrostatics!}$$

$$\Rightarrow \phi(\vec{r},t) = \int d^3r' \frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}$$

no matter what motion the source $\rho(\vec{r},t)$ has! ϕ is given by the instantaneous Coulomb potential even though electromagnetic fields have a finite velocity of propagation c!

Ampere's Law becomes:

$$-\nabla^2 A + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{j} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right)$$

$$\Rightarrow \Box^2 A = \frac{4\pi}{c}\vec{j} - \frac{1}{c}\vec{\nabla}\left(\frac{\partial\phi}{\partial t}\right)$$

where $\vec{\nabla}\left(\frac{\partial\phi}{\partial t}\right) = \vec{\nabla}\left[\int d^3r' \, \frac{\partial\rho}{\partial t}\,\frac{1}{|\vec{r}-\vec{r}'|}\right]$

$$= -\vec{\nabla}\left[\int d^3r' \, \frac{\vec{\nabla}'\cdot\vec{j}(\vec{r}',t)}{|\vec{r}-\vec{r}'|}\right]$$ by continuity eqn.

To see the meaning of this term, recall – any vector function $\vec{j}$ can be written as the sum of a curlfree and a divergenceless part

$$\vec{j} = \vec{j}_{\parallel} + \vec{j}_{\perp}$$ where $\vec{\nabla}\times\vec{j}_{\parallel} = 0$ curlfree

$\vec{\nabla}\cdot\vec{j}_{\perp} = 0$ divergenceless

where

$$\vec{j}_{\parallel}(\vec{r}) = -\frac{1}{4\pi}\vec{\nabla}\int d^3r' \, \frac{\vec{\nabla}'\cdot\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$ longitudinal part

$\vec{j}_{\perp}(\vec{r}) =$ transverse part

$$= \frac{1}{4\pi}\vec{\nabla}\times\int d^3r' \, \frac{\vec{\nabla}'\times\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

So $\vec{\nabla}\left(\frac{\partial\phi}{\partial t}\right) = 4\pi\vec{j}_{\parallel}$, and

$$\Box^2 A = \frac{4\pi}{c}\vec{j} - \frac{4\pi}{c}\vec{j}_{\parallel} = \frac{4\pi}{c}\vec{j}_{\perp}$$