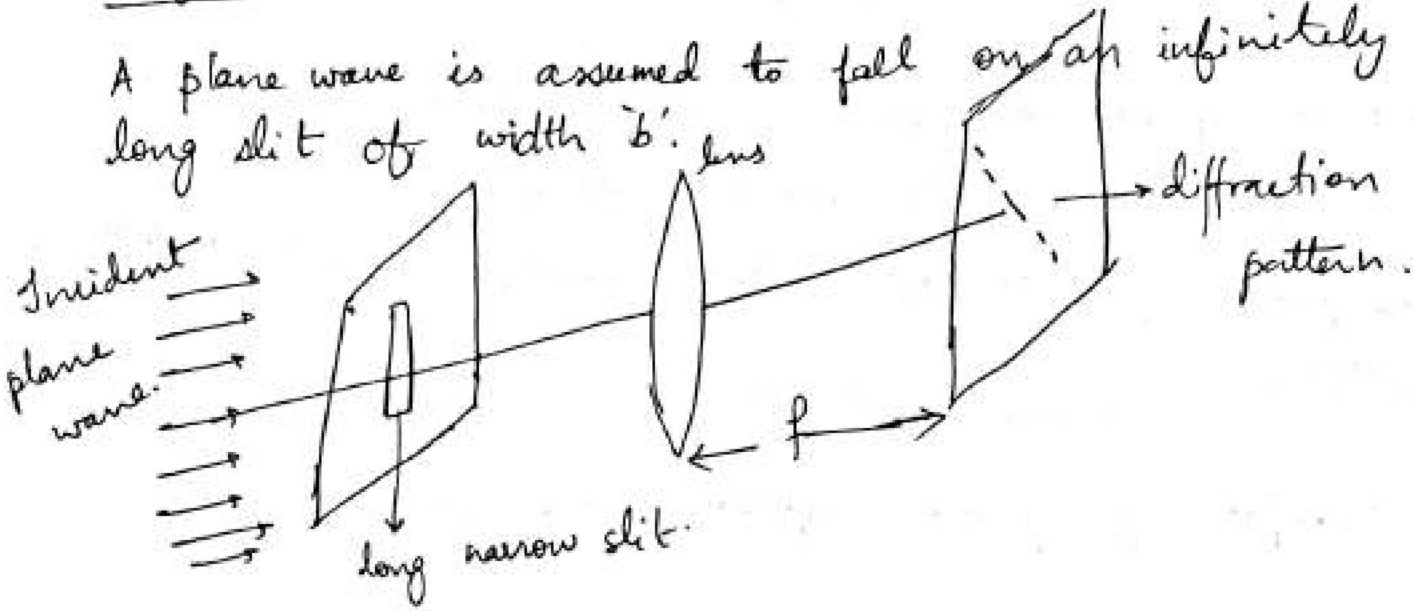
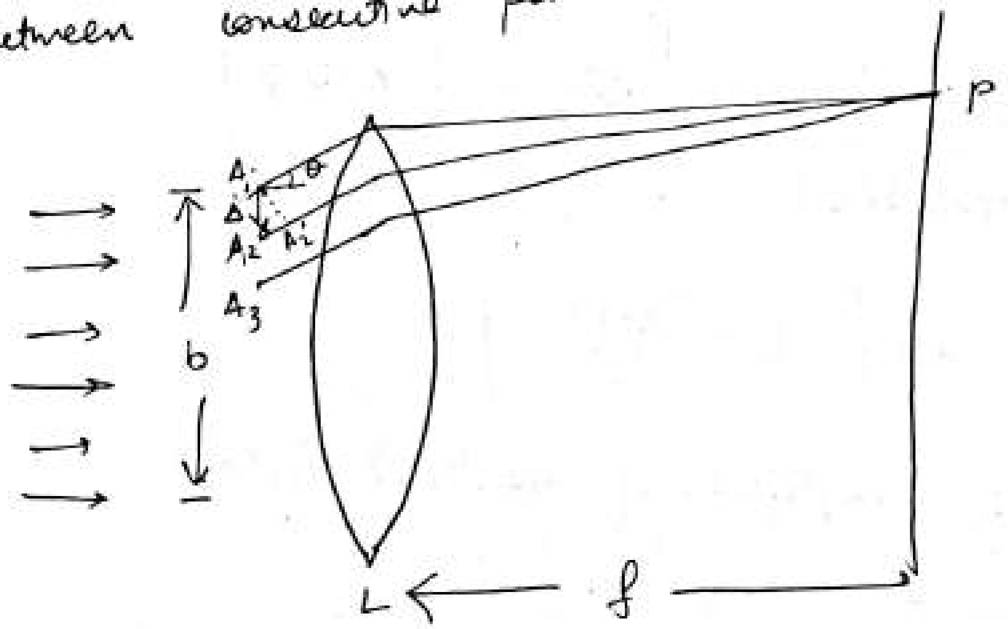
Intensity distribution in diffraction pattern due to a Single Slit

A plane wave is assumed to fall on an infinitely long slit of width 'b'.



To calculate the intensity distribution on the focal plane of lens L, assume that the slit consists of a large number of equally spaced point sources and each point on the slit is a source of Huygen's secondary wavelets which interfere with the wavelets coming from other points.

Let A_1, A_2, A_3 .. be the point sources and distance between consecutive points be Δ



∴ if number of point sources is n
then $b = (n-1)\Delta$.

Let P be a point on screen receiving parallel rays making an angle θ with the normal to slit.
The path difference b/w waves coming from A_1 & A_2 will be

$$A_2A_2' = \Delta \sin\theta$$

corresponding phase difference ϕ is given by

$$\phi = \frac{2\pi}{\lambda}\Delta\sin\theta$$

if field at P due to A_1 is $a\cos\omega t$
then field at P due to A_2 is $a\cos(\omega t-\phi)$.
∴ similarly phase diff. between waves coming from A_2 & A_3 will also be ϕ.

$$\therefore E = a\left[\cos\omega t + \cos(\omega t-\phi) + \ldots\ldots + \cos(\omega t-(n-1)\phi)\right]$$

where $\phi = \frac{2\pi}{\lambda}\Delta\sin\theta$

$$\therefore E = a\frac{\sin(n\phi/2)}{\sin(\phi/2)}\cos\left[\omega t - \frac{1}{2}(n-1)\phi\right]$$

$$E = E_0\cos\left[\omega t - \frac{(n-1)}{2}\phi\right]$$

where E_0 amplitude of resultant field is

$$E_0 = a\frac{\sin(n\phi/2)}{\sin(\phi/2)}$$

In the limit $n \to \infty$ and $\Delta \to 0$ $\quad n\Delta \to b$

we have

$$\frac{n\phi}{2} = \frac{n\pi}{\lambda}\Delta \sin\theta \to \frac{\pi b}{\lambda}\sin\theta \qquad (1)$$

$$\text{Hence} \quad \phi = \frac{2\pi}{\lambda}\Delta \sin\theta = \frac{2\pi b}{n\lambda}\sin\theta \qquad (2)$$

$$\cancel{\therefore\ E_0 = \frac{a\sin(n\phi/2)}{}}$$

$$\phi/2 = \frac{\pi\Delta\sin\theta}{\lambda} \to 0 \quad \text{as } n\to\infty \ \& \ \Delta \to 0.$$

$$\therefore \ \sin(\phi/2) \simeq \phi/2$$

$$\therefore \ E_0 = \frac{a\sin(n\phi/2)}{\phi/2}$$

$$= \frac{a\sin\left(\frac{\pi b\sin\theta}{\lambda}\right)}{\frac{\pi b\sin\theta}{\lambda n}} \quad (\text{from } (1) \ \& \ (2))$$

$$= na\,\frac{\sin\left(\frac{\pi b\sin\theta}{\lambda}\right)}{\frac{\pi b\sin\theta}{\lambda}}$$

$$= \frac{A\sin\beta}{\beta}$$

where $A = na$ & $\beta = \dfrac{\pi b\sin\theta}{\lambda}$.

$$\therefore \ E = \frac{A\sin\beta}{\beta}\cos(\omega t - \beta)$$

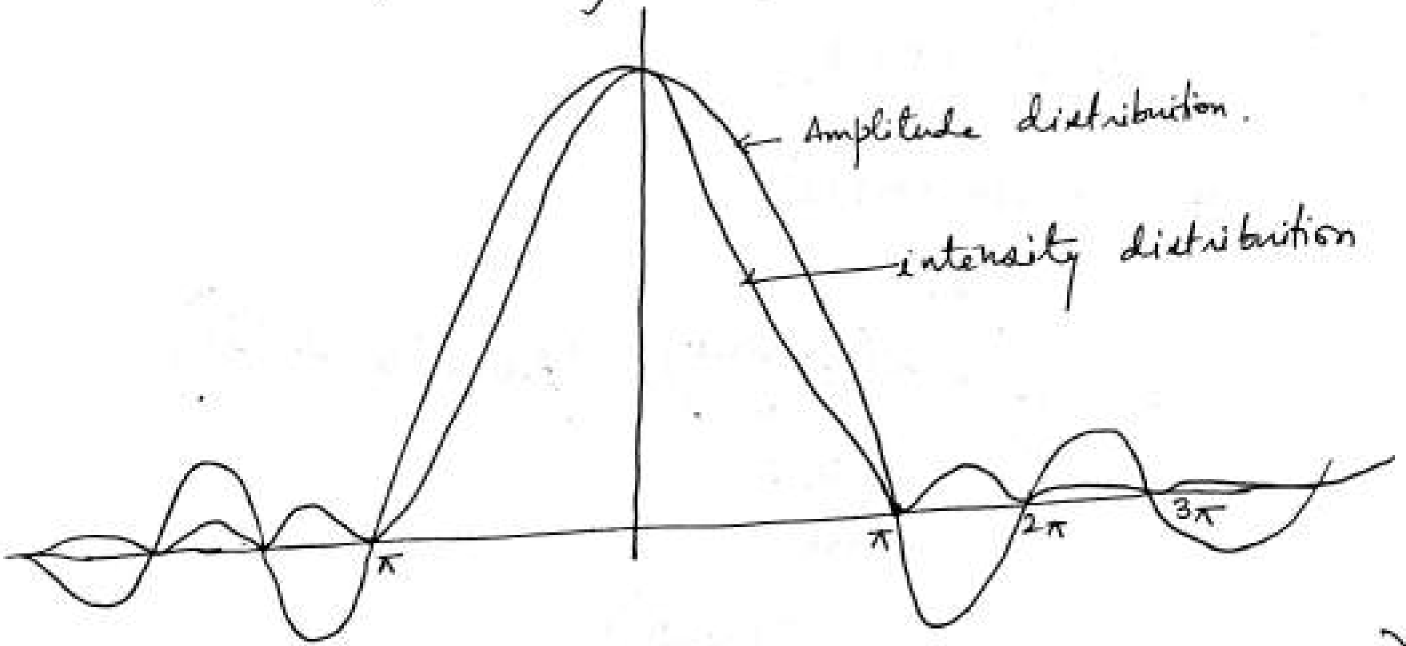
∴ corresponding intensity distribution is given by

$$I = I_0 \frac{\sin^2\beta}{\beta^2} \quad -(2)$$

where I_0 represents the intensity at $\theta = 0$

Position of maxima & minima.

Variation of intensity with β is shown in figure below



$I = 0$ when $\beta = m\pi$ & $m \neq 0$ (from eqn (3))

when $\beta = 0$ then $\frac{\sin\beta}{\beta} = 1$ & $I = I_0$ which correspond to max intensity.

$$\therefore \beta = \frac{\pi b \sin\theta}{\lambda} = m\pi$$

$$\Rightarrow b\sin\theta = m\lambda \qquad m = \pm 1, \pm 2, \pm 3 \ldots \text{(minima)}$$

To determine the positions of maxima we differentiate eq^n ③ w.r.t β.

$$\frac{dI}{d\beta} = I_0\left[\frac{2\sin\beta\cos\beta}{\beta^2} - \frac{2\sin^2\beta}{\beta^3}\right] = 0$$

$$\Rightarrow \quad \frac{\sin\beta\cos\beta}{\beta^2} = \frac{\sin^2\beta}{\beta^3}$$

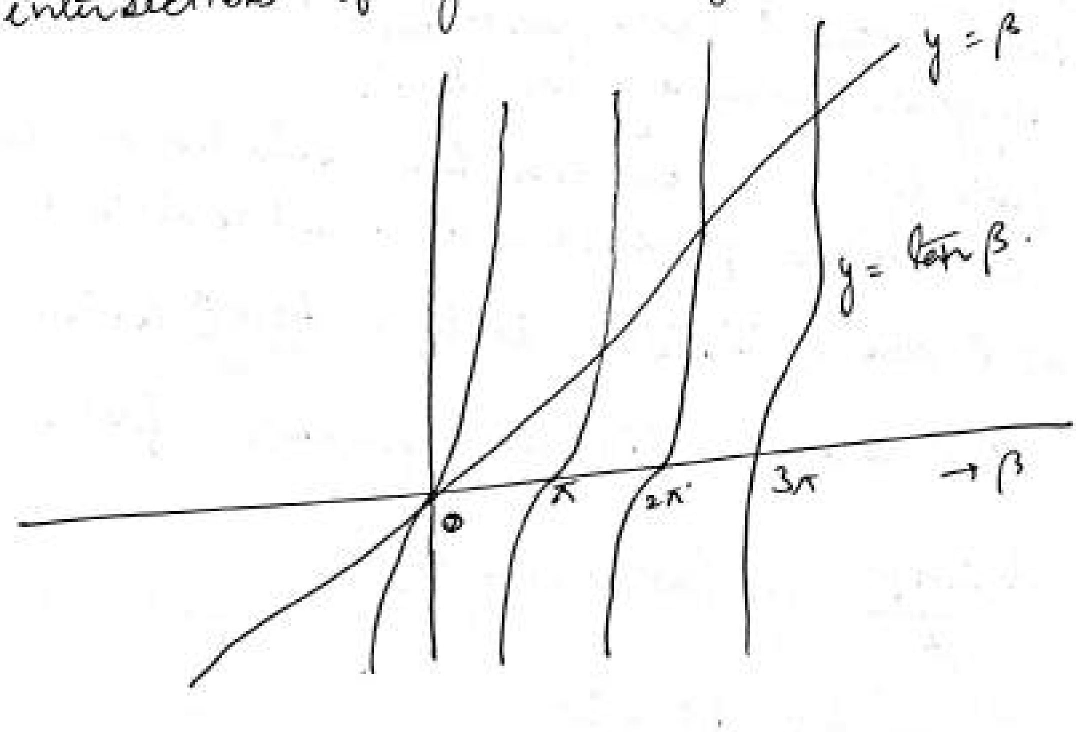
$$\Rightarrow \quad \beta\cos\beta = \sin\beta$$

$$\Rightarrow \quad \cancel{\beta = \tan} \Rightarrow \tan\beta = \beta.$$

The conditions for maxima are roots of the transcendental equation $\tan\beta = \beta$ (maxima).

$\beta = 0 \Rightarrow$ central maxima.

The other roots can be obtained by determining the points of intersections of $y = \beta$ and $y = \tan\beta$.

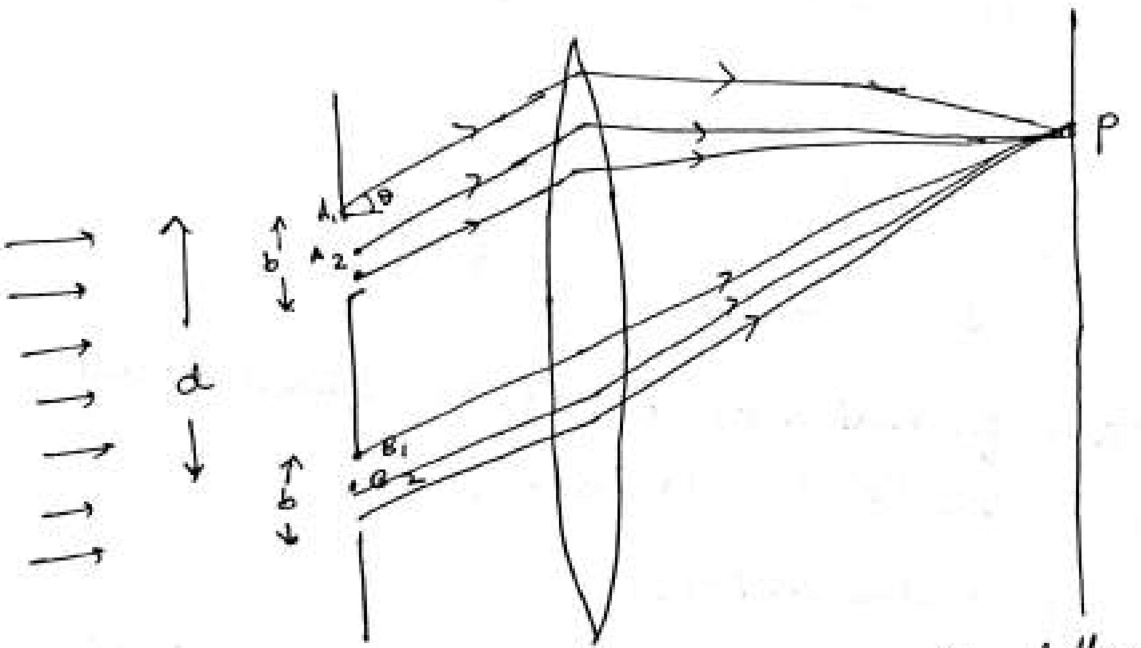


The intersections occur at

$\beta = 1.43\pi$ first maxima

$\beta = 2.46\pi$ etc. second maxima

Fraunhoffer Diffraction at Double Slit



Similar to the approach in single slit diffraction we assume that the slits consist of a large number of equally spaced point sources & each point on the slit is a source of Huygen's secondary wavelets.

Hence the path difference between the disturbances reaching P from two consecutive points in a slit will be $\Delta \sin\theta$

$\therefore$ field at P due to I slit is $E_1 = \frac{A\sin\beta}{\beta}\cos(\omega t - \beta)$

Similarly the second slit will produce a field

$$E_2 = \frac{A\sin\beta}{\beta}\cos(\omega t - \beta - \phi_1)$$

where $\phi_1 = \frac{2\pi}{\lambda} d\sin\theta$

ϕ_1 represents the phase difference between the disturbances from 2 corresponding points on the slits. By corresponding we mean pairs of points such that $A_1 B_1, A_2 B_2 \ldots$ which are separated by 'd'.

∴ resultant field at P will be

$$E = E_1 + E_2$$
$$= \frac{A \sin\beta}{\beta}\left[\cos(\omega t - \beta) + \cos(\omega t - \beta - \phi_1)\right]$$

$$E = 2A \frac{\sin\beta}{\beta} \cos\frac{\phi_1}{2} \cos\left(\omega t - \beta - \frac{\phi_1}{2}\right)$$

let $\gamma = \frac{\phi_1}{2} = \frac{\pi d}{\lambda}\sin\theta$

The intensity distribution will be of form

$$I = 4 I_0 \frac{\sin^2\beta}{\beta^2} \cos^2\gamma \quad -(4)$$

represents intensity distribution of diffraction by a single slit of width b.

represents the interference pattern produced by two point sources separated by 'd'.

From eqn (4)

intensity is zero whenever

$\beta = \pi, 2\pi, 3\pi \ldots$

or when $\gamma = \pi/2, 3\pi/2, 5\pi/2 \ldots$

The corresponding angles of diffraction are given by the following equations.

$$b\sin\theta = m\lambda \qquad m = 1, 2, 3 ..$$

$$d\sin\theta = (n + 1/2)\lambda \qquad n = 1, 2, 3 ..$$

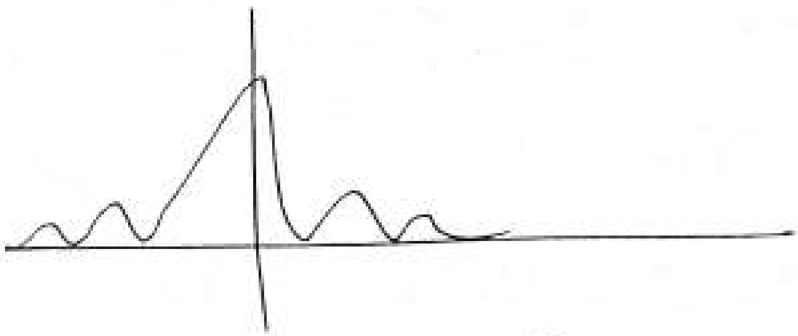
interference maxima ~~to~~ occur when

$$\gamma = 0, \pi, 2\pi, ...$$

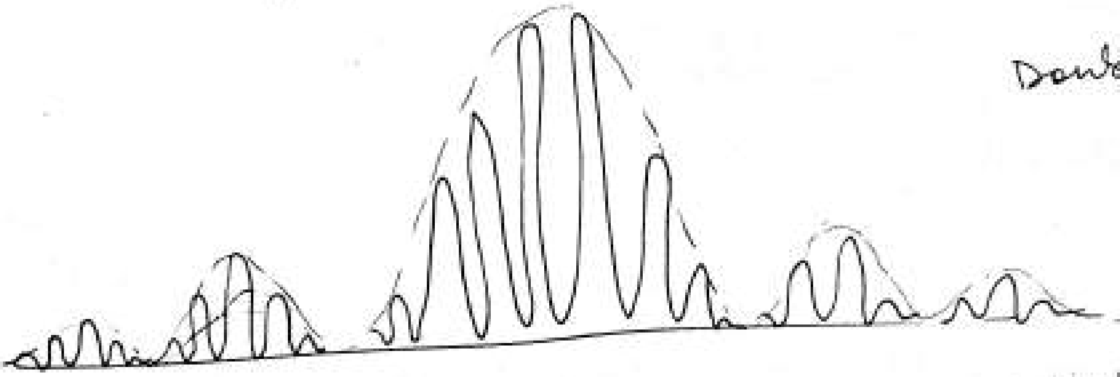
or when

$$d\sin\theta = 0, \lambda, 2\lambda, 3\lambda ...$$

Distinction b/w single slit & double slit diffraction patterns



single slit pattern



Double slit pattern.

Dotted lines represent the intensity distribution due to diffraction pattern whereas the solid line represents the intensity distribution due to interference b/w the light from both the slits.

Double slit pattern consists of equally spaced interference maxima & minima with in the central maximum.

The spacing of diffraction max & min depend on 'a'. spacing of interference max & min depend on a & b

Missing orders in a double slit diffraction pattern.

Depending on the relative values of 'a' & 'b' certain orders of interference maxima will be missing in the resultant pattern.

Direction of interference maxima are given by

$(a+b)\sin\theta = n\lambda$

The direction of diffraction minima are given by

$$a\sin\theta = p\lambda .$$

n & p are integers.

if the values of 'a' & 'b' are such that both the equations are satisfied simultaneously for the same value of θ, then the position of certain interference maxima correspond to the diffraction minima at the same position on the screen.

let $a = b$

then $2a\sin\theta = n\lambda$

& $a\sin\theta = p\lambda$

i.e $\frac{n}{p} = 2$ or $n = 2p$

i.e $p = 1, 2, 3,$ $n = 2, 4, 6$ etc

Thus the orders 2, 4, 6 etc of interference maxima will be missing in the diffraction pattern.

if $2a = b$.

then $3a\sin\theta = n\lambda$

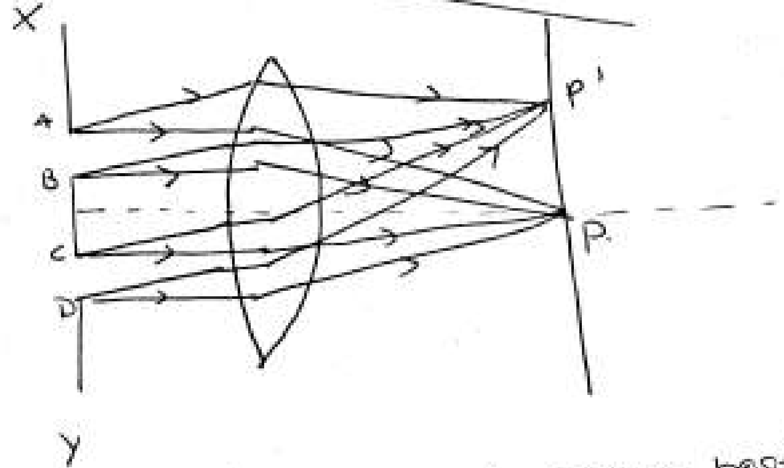
$a\sin\theta = p\lambda$

$\frac{n}{p} = 3.$

$p = 1, 2, 3$ & $n = 3, 6, 9$ etc. 3, 6, 9 will be missing in the diffraction pattern.

The position of ~~first~~ third ~~diffraction~~ interference max. correspond to first diffraction minimum.

Diffraction at Double Slit

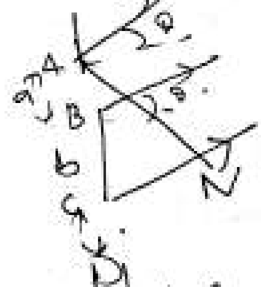


Width of each slit → a & opaque portion is b'.

diffraction pattern is due to

(i) the interference phenomenon due to the secondary waves emanating from the corresponding points of the 2 slits &

(ii) the diffraction pattern due to the secondary waves from the slits individually.



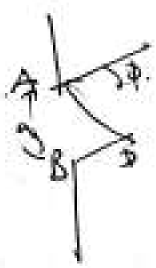
Interference maxima & minima.

$$CN = (a+b)\sin\theta$$

If this path difference is equal to odd multiple of $\lambda/2$, a gives the direction of minima due to the interference of the secondary waves from the 2 slits.

$$CN = (a+b)\sin\theta_n = (2n+1)\lambda/2$$

θ_n → location of minima.



Diffraction maxima & minima.

$$BD = a\sin\phi = n\lambda \rightarrow \text{minima}$$