Power Series

Def: A series of the form

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \equiv \sum_{n=0}^{\infty} a_nx^n$$

is called a <u>power series</u> in x, ~~and the~~

The numbers a_n (dependent on <u>n</u> but not on x)

their coefficients.

<u>Note</u>: For $x=0$, every power series is convergent

whatever be the value of the coeff.

Most Imp Facts about a P.S. is that

<u>either</u>

(i) it converges for no value of x other than $x=0$, we then say that it is <u>nowhere convergent</u>.

e.g. $\sum n^n x^n$ converges for no value of x, other than $x=0$

<u>Or</u> it converges for all values of x.

Then it is called <u>everywhere convergent</u>

e.g. $\sum \frac{x^n}{n}$, $\sum (-1)^n \frac{x^n}{n!}$

or (the general case) it converges for some values of x and diverges for others —

def

Region of convergence:

The totality of points x for which it converges is called its region of convergence.

Radius of convergence: A set of values for which a P.S. converges is called radius of convergence

A definite +ve no. R ($R \geq 0$) s.t.

$$\sum a_n x^n \text{ converges for } |x| < R$$

$$\text{diverges for } |x| > R$$

is called the radius of convergence of the series.

The interval $|x| < R$ or $-R < x < R$ is called the interval of convergence.

If $\overline{lt}\ |a_n|^{1/n} = \frac{1}{R}$, then the series $\sum a_n x^n$ is convergent (absolutely) for $|x| < R$

diverges for $|x| > R$.

$$\overline{lt}_{n \to \infty} |a_n x^n|^{1/n} = \frac{|x|}{R} \quad \left(\because \overline{lt}\ |a_n|^{1/n} = \frac{1}{R}\right)$$

By the Cauchy's root test, the series $\sum a_n x^n$ is absolutely convergent &

$\therefore$ convergent for $|x| < R$ $\quad \left(\frac{|x|}{R} < 1\right)$

& divergent for $|x| > R$.

$\frac{|x|}{R} > 1$

How to Calculate Radius of Convergence

The radius of convergence R of a P.S. is defined to be equal to

$$R = \frac{1}{\overline{lt}\, |a_n|^{1/n}} \quad \text{when } \overline{lt}\, |a_n|^{1/n} > 0$$

$$\infty \quad \text{when } \overline{lt}\, |a_n|^{1/n} = 0$$

$$0 \quad \text{when } \overline{lt}\, |a_n|^{1/n} = \infty.$$

→ The radius of convergence can also be found by the relation

$$R = \underset{n \to \infty}{lt} \left| \frac{u_n}{u_{n+1}} \right|$$

provided the lt. exists

Note: For a nowhere convergent P.S. $R = 0$

for everywhere convergent P.S $R = \infty$

Thm: If a P.S. $\sum a_n x^n$ converges for $x = x_0$ then it is absolutely convergent for every $x = x_1$ when $|x_1| < |x_0|$

Pf: Since the series $\sum a_n x_0^n$ is convergent

$\therefore$ $a_n x_0^n \to 0$ as $n \to \infty$ (Nece. con. of conv. of series)

$\therefore$ for $\epsilon = \frac{1}{2}$, $\exists$ an int. N s.t.

$$|a_n x_0^n - 0| < \frac{1}{2} \quad \text{for } n \geq N$$

and
$$|a_n x_1^n| = |a_n x_0^n| \cdot \left|\frac{x_1}{x_0}\right|^n$$
$$< \frac{1}{2}\left|\frac{x_1}{x_0}\right|^n \quad \text{for } n \geq N$$

But $\sum \left|\frac{x_1}{x_0}\right|^n$ is a convergent geo. series with common Ratio $\left|\frac{x_1}{x_0}\right| < 1$.

$\therefore$ by comparison test, the series $\sum |a_n x_1^n|$ converges.

Hence $\sum a_n x^n$ is absolutely convergent for every $x = x_1$ when $|x_1| < |x_0|$