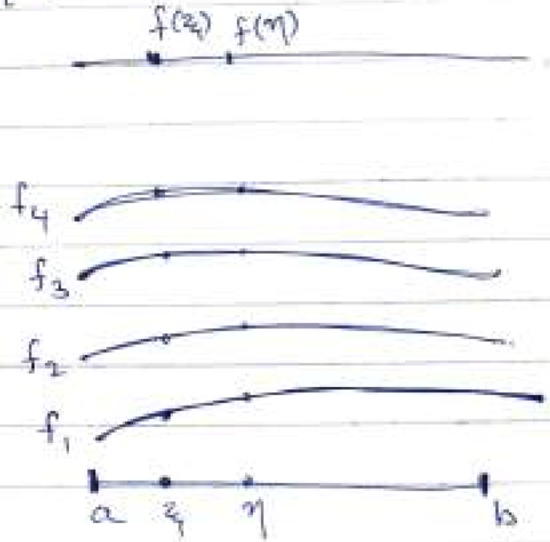
Sequences & Series of fns
(Pointwise and Uniform Convergence)

.: If to every natural no. n, $\exists$ a fn f_n, then the family of fns $\{f_n\}$ is called a sequence of fns.

The fns f_n may have diff. domains or the same domains.

We discuss the fns having same domain.



uppose $\{f_n\}$, $n = 1, 2, 3 \ldots$ is a seq of fns
efined on an interval I, $a \le x \le b$

o each $\xi \in I$, $\exists$ a seq. of nos $\{f_n(\xi)\}$

ith terms $f_1(\xi), f_2(\xi), f_3(\xi)$.

t us suppose that the seq. of nos $\{f_n(\xi)\}$.
nverges for every $\xi \in I$

et $\{f_n(\xi)\}$ converge to $f(\xi)$

his way let the sequences at all pts $\xi, \eta, \ldots$ of I converge
$f(\xi)$, $f(\eta)$, ...

For a seq $f_n(x)$ of real fns having the common domain D at every pt. of which $f_n(x)$ converges, we define a fn $f(x)$ as follows:

$$f(x) = \lim_{n \to \infty} f_n(x) \quad \text{for every } x \in D.$$

Thus fn $f(x)$ is called the pt. wise limit of the sequence $f_n(x)$ and this convergence is called the pointwise convergence of $f_n(x)$ to $f(x)$.

Similarly, if the series Σf_n converges for every pt $x \in I$, & we define

$$f(x) = \sum_{n=0}^{\infty} f_n(x) \quad \forall \ x \in [a, b].$$

the fn f is called the sum or the pointwise sum of the series Σf_n on $[a, b]$.

Def: (Pointwise Limit)
If f is the pointwise limit of a sequence of fns $\{f_n\}$ defined on $[a, b]$, then to each $\epsilon > 0$ and to each $x \in [a, b]$, there correspon[ds] an integer m s.t.

$$|f_n(x) - f(x)| < \epsilon \quad \forall \ n \geq m.$$

If a sequence converges pointwise to f then for a given $\epsilon > 0$, each point x_i of $[a,b]$ determines an integer N_i s.t.

$$|f_n(x_i) - f(x_i)| < \epsilon \quad \forall \; n \geq N_i$$

Consideration of all pts of $[a,b]$ gives rise to a seq. of integers $N_1, N_2, N_3 \cdots \cdots$

In case the sequence $\{N_i\}$ is bounded above, with supremum N, say, then

$$|f_n(x) - f(x)| < \epsilon \quad \forall \; n \geq m \qquad \text{———} \; (*)$$

holds for all pts. of $[a,b]$ when $n \geq N$ and so the given sequence $\{f_n\}$ converges uniformly on $[a,b]$.

Difference in Pt wise & uniform convergence (U.C.):

In case of pt-wise convergence, for each $\epsilon > 0$ & for each $x \in [a,b]$ $\exists$ an integer N (depending on ϵ and x both) s.t. $(*)$ holds for $n \geq N$.

In U.C., $\forall \; \epsilon > 0$, it is possible to find one integer N (depending on ϵ alone) which will do $\forall \; x \in [a,b]$.

$\to$ Uniform convergence $\Rightarrow$ pt. wise convergence but not vice versa.

$\Rightarrow$ Non point wise convergence $\Rightarrow$ non-uniform convergence.

i.e. a fn which is not pt. wise convergent cannot be uniformly convergent.

Uniform Convergence

A sequence of fns $\{f_n\}$ is said to converge uniformly on an interval $[a,b]$ to a fn f if for any $\epsilon > 0$ and for all $x \in [a$ $\exists$ an integer N (independent of x but dependent on ϵ) s.t. for all $x \in [a,b]$

$$| f_n(x) - f(x) | < \epsilon \quad \forall\ n \geq N.$$

Every uniformly convergent seq. is pointwise convergent & the uniform limit fn is same as the point wise limit f

* A series of fns $\sum f_n$ is said to converge uniformly on $[a,b]$ if the sequence $\{S_n\}$ of its partial sums, defined by

$$S_n(x) = \sum_{i=1}^{\infty} f_i(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$$

converges uniformly on $[a,b]$.

→ A series of fns $\sum f_n$ converges uniformly to f on $[a,b]$ if for $\epsilon > 0$ and all $x \in [a,b]$, $\exists$ an integer N (independent of x and dependent on ϵ) s.t. for all x in $[a,b]$

$$|f_1(x) + f_2(x) + \cdots + f_n(x) - f(x)| < \epsilon \quad \text{for } n \geq N$$

Cauchy's Criterion for uniform convergence:

sequence of fns $\{f_n\}$ defined on $[a,b]$ converges uniformly on $[a,b]$ iff for every $\epsilon > 0$ & $\forall x \in [a,b]$, $\exists$ an integer N such that

$$|f_{n+p}(x) - f_n(x)| < \epsilon \quad \forall n \geq N, \ p \geq 1.$$

Q. S.T. the seq.

$$f_n(x) = x^n$$

is unif. conv. on $[0,k]$, $k<1$ & only pt. wise convergent on $[0,1]$

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & , 0 \le x < 1 \\ 1 & x = 1 \end{cases}$$

Thus the seq. converges pt. wise to a discontinuous fn on $[0,1]$.

Let $\epsilon > 0$ be given.

For $0 < x < 1$, we have

$$|f_n(x) - f(x)| = x^n < \epsilon$$

if $\left(\frac{1}{x}\right)^n > \frac{1}{\epsilon}$

if $\log\left(\frac{1}{x}\right)^n > \log 1/\epsilon$

if $n > \dfrac{\log 1/\epsilon}{\log 1/x}$

Let N be a natural no. $\geq \dfrac{\log 1/\epsilon}{\log 1/k}$

$k > 0$ in $]0,k]$

$\therefore \; |f_n(x) - f(x)| < \epsilon \quad \forall\, n \geq N,\ 0<x<1$

For $x = 0$

$$|f_n(x) - f(x)| = 0 < \epsilon \quad \forall \ n \geq 1$$

Thus for any $\epsilon > 0$ $\exists$ N s.t. $\forall$ $x \in [0, k]$, $k < 1$

$$|f_n(x) - f(x)| < \epsilon \quad \forall \ n \geq N.$$

Note: the no. $\dfrac{\log 1/\epsilon}{\log 1/x} \longrightarrow \infty$ as $x \longrightarrow 1$

So it is not possible to find an integer N s.t.

$$|f_n(x) - f(x)| < \epsilon \quad \forall \ n \geq N \ \& \ \forall \ x \in [0, 1]$$

Hence the seq. is not unif conv. on any interval containing 1 & in particular on $[0, 1]$.