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Q: Check the converg.

$$\sum_{n=1}^{\infty} \frac{1\cdot 2\cdot 3 \cdots n}{7\cdot 10 \cdots (3n+4)}$$

$$u_n = \frac{1\cdot 2\cdot 3 \cdots n}{7\cdot 10 \cdots (3n+4)}$$

$$u_{n+1} = \frac{1\cdot 2\cdot 3 \cdots n(n+1)}{7\cdot 10 \cdots (3n+4)(3(n+1)+4)}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{3n+7}{n+1} = \lim_{n\to\infty} \frac{3+7/n}{1+1/n} = 3 > 1$$

$\therefore$ By Ratio Test, series converges.

Q: $\frac{x}{1\cdot 3} + \frac{x^2}{2\cdot 4} + \frac{x^3}{3\cdot 5} + \frac{x^4}{4\cdot 6} + \cdots \quad (x>0)$

$$u_n = \frac{x^n}{n(n+2)} \qquad u_{n+1} = \frac{x^{n+1}}{(n+1)(n+3)}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1}{x}\left(\frac{n+1}{n}\right)\left(\frac{n+3}{n+2}\right) = \frac{1}{x}$$

By Ratio Test

$\sum u_n$ converges if $\frac{1}{x} > 1$ i.e $x < 1$

$\sum u_n$ div. if $\frac{1}{x} < 1$ i.e $x > 1$

Test fails for $x = 1$

For $x = 1$, $u_n = \frac{1}{n(n+2)} \sim \frac{1}{n^2}$ (for large nos of n

Let $v_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n}{n+2} = 1$$

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So $\sum U_n$ & $\sum V_n$ converge or div. together

$\because$ $\sum v_n = \sum \frac{1}{n^2}$ converges, so $\sum u_n$ conv. for $x=1$

$\therefore$ Finally the given series conv. for $x \leq 1$

& div for $x > 1$

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Alternating Series :

A series of the form
$\sum_{n=1}^{\infty}(-1)^{n-1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$ where $u_n > 0 \ \forall\ n \in \mathbb{N}$
is called an Alternating Series

Leibnitz Test

If an alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ satisfies

(i) $u_{n+1} \leq u_n \ \forall n$

(ii) $\lim_{n\to\infty} u_n = 0$

then the series $\sum(-1)^{n-1}u_n$ converges.

Pf:

To prove : The series converges
i.e. to prove Its seq. of partial sums converges.

$S_{2n} = u_1 - u_2 + u_3 - u_4 + \cdots + u_{2n-1} - u_{2n}$

$S_{2n+2} = u_1 - u_2 + u_3 - u_4 + \cdots + u_{2n-1} - u_{2n} + u_{2n+1} - u_{2n+2}$

$\therefore\ S_{2n+2} - S_{2n} = u_{2n+1} - u_{2n+2} \geq 0 \quad (\because u_{n+1} \leq u_n \ \forall n)$

Thus $\langle S_{2n} \rangle$ is a monotonically increasing seq. — ①

$S_{2n} = u_1 - u_2 + u_3 - u_4 + u_5 - \cdots + u_{2n-1} - u_{2n}$
$= u_1 - [(u_2 - u_3) + (u_4 - u_5) + \cdots + (u_{2n-2} - u_{2n-1}) + u_{2n}]$

Now $S_{2n} < u_1 \ \forall n$ $\quad \because u_{n+1} \leq u_n \ \forall n,\ u_{2n} > 0$

$\therefore \langle S_{2n} \rangle$ is bdd above. — ②

From ① & ②

$\langle S_{2n} \rangle$ is mono. incre. and bdd above.

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$\therefore \langle S_{2n} \rangle$ is cgt.

Let $\lim_{n\to\infty} S_{2n} = l$ ——— (3)

$S_{2n+1} = u_1 - u_2 + u_3 - \dots - u_{2n} + u_{2n+1}$

$= S_{2n} + u_{2n+1}$

$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + u_{2n+1}$

$= l + 0$

$\therefore \lim S_{2n+1} = l$ ——— (4)

From (3) & (4)

For any $\epsilon > 0$, $\exists$ natural nos. m_1 & m_2 s.t.

$|S_{2n} - l| < \epsilon \quad \forall \quad n \geq m_1$ ——— (5)

$|S_{2n+1} - l| < \epsilon \quad \forall \quad n \geq m_2$ ——— (6)

Let $m = \max(m_1, m_2)$ ——— (7)

From (5), (6) & (7)

$|S_n - l| < \epsilon \quad \forall \quad n \geq m$.

$\Rightarrow \langle S_n \rangle$ converges to l.

$\therefore \sum_{n=1}^{\infty} (-1)^{n-1} u_n$ converges.

Q: Check the convergence of the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots$$

Pf : (i) $n+1 > n \quad \forall n \in \mathbb{N}$

$$\frac{1}{n+1} < \frac{1}{n}$$

$\therefore \quad u_{n+1} < u_n \quad \forall n.$

(ii) $\lim_{n \to \infty} \frac{1}{n} = 0$

$\therefore$ By Leibnitz's test, the given series converges.