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Linear Transformations

See Sec 5.1 first page in the book for the definition of Domain, codomain range, image and preimage in the content of a function.

Def Let V and W be vector spaces and $f: V \to W$ be a function. Then f is a linear transformation iff both the conditions are satisfied

1. $f(v_1+v_2) = f(v_1)+f(v_2) \quad \forall\, v_1, v_2 \in V$
2. $f(cv) = cf(v) \quad \forall\, c \in R,\ v \in V$.

e.g. 1 $f: M_{m\times n} \to M_{n\times m}$ s.t.
$f(A) = A^T \quad \forall\, A \in M_{m\times n}$

1. $f(A+B) = (A+B)^T = A^T + B^T = f(A)+f(B)$
2. $f(aA) = (aA)^T = aA^T = af(A) \quad \forall\, a \in R$

Hence f is a L.T.

Example 2 $f: P_n \to P_{n-1}$
$f(P) = \underbrace{P'}_{\text{derivative of } P}$ where $P \in P_n \to$ Poly. of degree at most n

1. $f(P_1+P_2) = (P_1+P_2)' = P_1' + P_2' = f(P_1)+f(P_2)$
2. $f(aP) = (aP)' = aP' = af(P) \quad \forall\, a \in R \quad \forall\, P \in P_n$

$\Rightarrow f$ is a L.T.

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example ② Let V be a v.s. of dimension n. ~~de~~ and $\beta = \{\beta_1, \beta_2, \ldots \beta_n\}$ is basis of V. Then for every vector $v \in V$, v is linear combination of elements of β. i.e $\exists$ scalars $a_1, a_2, \ldots a_n$ such that $v = a_1\beta_1 + a_2\beta_2 + \ldots a_n\beta_n$

define $[v]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

$[v]_\beta$ is called coordinate vector of v w.r.t. Basis β.

e.g. $V = \mathbb{R}^2$ take $\beta = \{(1,1), (1,0)\}$

let $v = (3,2)$ then

$(3,2) = 2(1,1) + (1,0)$

$\Rightarrow [(3,2)]_\beta = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

lly $(0,1) = (1,1) - (1,0)$

so $[(0,1)]_\beta = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

i.e for every $v \in V$ & fixed basis β $[v]_\beta$ is unique ~~and~~ v.

If we change basis β then $[v]_\beta$ is different.

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Now define $f : V \to \mathbb{R}^n$ s.t.

$$f(v) = [v]_\beta$$

where β is a fixed basis of V.
Then

① $f(v_1+v_2) = [v_1+v_2]_\beta$
$= [v_1]_\beta + [v_2]_\beta$
$= f(v_1) + f(v_2)$

② $f(av) = [av]_\beta = a[v]_\beta = af(v)$

$\Rightarrow f$ is a L.T.

Example ④ $f : \mathbb{R}^2 \to \mathbb{R}^2$ s.t.
$f(x,y) = (x+1, y+1)$

① $f(v_1+v_2) = f((x_1,y_1)+(x_2,y_2))$
$= f(x_1+x_2, y_1+y_2)$
$= (x_1+x_2+1, y_1+y_2+1)$

and
$f(v_1)+f(v_2) = f(x_1,y_1)+f(x_2,y_2)$
$= (x_1+1, y_1+1)+(x_2+1, y_2+1)$
$= (x_1+x_2+2, y_1+y_2+2)$

$\Rightarrow f(v_1+v_2) \neq f(v_1)+f(v_2)$

$\Rightarrow f$ is not a L.T.

Linear Operator (L.O.)

A linear transformation whose domain and codomain are same vector spaces

e.g. ① $I : V \to V$ s.t.
$I(v) = v \quad \forall\, v \in V$
$I \to$ Identity mapping from V to V
① $I(v_1 + v_2) = v_1 + v_2 = I(v_1) + I(v_2)$
② $I(av) = av = aI(v) \quad \forall\, a \in \mathbb{R}\ \&\ v \in V$.

$\Rightarrow I$ is a Linear operator

eg ② $O : V \to V$ s.t.
$O(v) = 0 \to$ zero of $V \quad \forall\, v \in V$
ie O is zero mapping i.e it takes every element to zero.
O is a linear operator.

e.g. ③ $f : \mathbb{R}^3 \to \mathbb{R}^3$ s.t.
$f(a_1, a_2, a_3) = (a_1, a_2, -a_3)$

is a L.O. you can check easily
It is called reflection through xy plane

eg. ④ $f : \mathbb{R}^n \to \mathbb{R}^n$ s.t.
$f(v) = kv$, k is a scalar
check that f is a L.O.
If $|k| > 1$ then f is called dilation & if $|k| < 1$ then f is called contraction.

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Ex 5) $f: R^3 \to R^3$ s.t.
$f(a_1, a_2, a_3) = (a_1, a_2, 0)$
This mapping takes elts of from XYZ plane to XY plane
check that it is a L.O.
This mapping is called a Projection

Ex 6) $f: R^n \to R^m$ s.t.
$f(x) = Ax \quad \forall x \in R^n$
where A is an $m \times n$ matrix.

$f(x_1 + x_2) = A(x_1 + x_2) = Ax_1 + Ax_2 = f(x_1) + f(x_2)$
& $f(ax) = A(ax) = aAx = af(x)$
$\Rightarrow f$ is a L.T.
If $n = m$ then f is L.O.

Thm 5.1 Let V & W are V.S. & $L: V \to W$ be a linear transformation. Then
(1) $L(0_V) = 0_W$
means zero vector of V will map to zero vector of W under the L.T.

(2) $L(-v) = -L(v)$
means additive inverse of v will map to additive inverse of $L(v)$

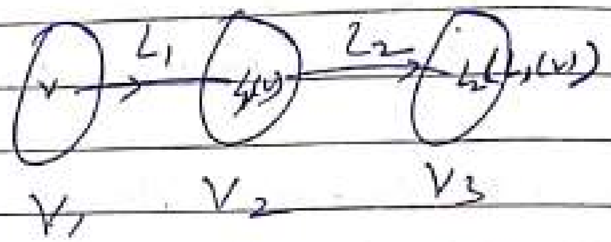
(3) $L(a_1 v_1 + \cdots + a_n v_n) = a_1 L(v_1) + a_2 L(v_2) + \cdots + a_n L(v_n)$
means image under a L.T. of a linear combination of vectors is linear combination of the images.

Proof see from the book.

<u>Thm 5.2</u> V_1, V_2, V_3 are v.s. $L_1 : V_1 \to V_2$ and $L_2 : V_2 \to V_3$ are L.T. Then $L_2 \circ L_1 : V_1 \to V_3$

~~and $L_1 \circ L_2 : V_2 \to$~~

given by $L_2 \circ L(v) = L_2(L_1(v))$ is a L.T.



Proof see from the book.

<u>Example</u> $L_1 : \mathbb{R}^2 \to \mathbb{R}^2$ s.t.

$$L_1(v) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

and $L_2 : \mathbb{R}^2 \to \mathbb{R}^2$ s.t.

$$L_2(x, y) = (-x, y)$$

so $L_2 \circ L_1 : \mathbb{R}^2 \to \mathbb{R}^2$ s.t.

$$L_2 \circ L_1(x, y) = L_2 \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= L_2 \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} -x\cos\theta + y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

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Linear Transformation & Subspaces

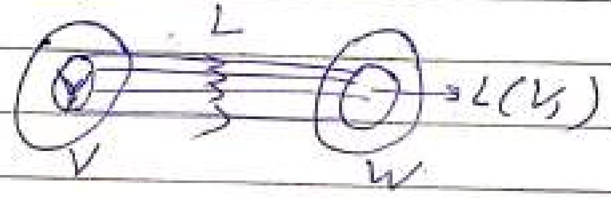
V & W are V.S.

Thm 5.3 Let $L: V \to W$ be a L.T. Then,

① If V_1 is subspace of V then $L(V_1) = \{L(v) \mid v \in V_1\}$ is subspace of W.

means image of a subspace under a L.T. is a subspace as $L(V_1) \to$ images of vectors of V_1 i.e. image of V_1

~~② If V_1 is sub~~

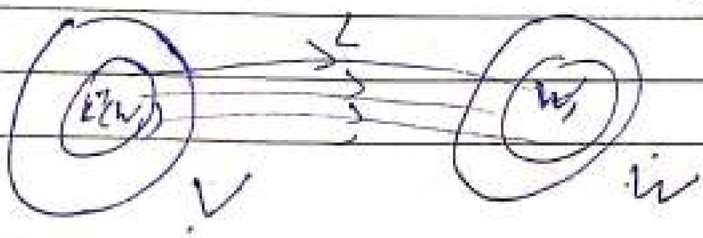


V_1 is subspace of V then $L(V_1)$ is subspace of W.

② If W_1 is subspace of W then $L^{-1}(W_1) = \{v \mid L(v) \in W_1\}$ is subspace of V

means if W_1 is subsp. of W then its preimage forms a subspace of V under the L.T. as

$L^{-1}(W_1) = \{L^{-1}(w) \mid w \in W_1\}$



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By (1) we get that $L(V)$ is subspace of W. i.e ~~[illegible]~~ Range of L is subspace of W.

~~By (2) we get that~~

~~Do questions from~~

example $L : M_{22} \to R^3$ where

$$L\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (b, 0, c)$$

verify that L is a L.T.

So by (1) of Thm 5.3 the range of L i.e $\{(b,0,c) \mid b, c \in R\}$ is subspace of R^3.

Also $V_1 = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, d \in R \right\}$ is subspace of M_{22}

So $L(V_1)$ is subspace of R^3

and $L(V_1) = \{(b,0,0) \mid b \in R\}$ is subspace of R^3.

~~Also $W_1 = \{(a, b, 2c) \mid a, b \in R\}$ is subspace of R^3 so its preimage will be subsp. of M_{22}~~

~~& $L^{-1}(W_1) = \{ \begin{pmatrix} a & b \\ [illegible] \end{pmatrix}$~~

Section 5.1 exercise do upto Q.14