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5.2 The matrix of a Linear Transformation

Let $L: V \to W$ where V & W are finite dim. Then for fixed basis for V & W we can find a matrix associated with L. If we change ~~the bases~~ any of the bases the matrix will change.

A L.T. is determined by its action on a basis. If we know the values of a L.T. $L: V \to W$ on the basis of V the values of L can be determined for all the elements of v.s. V.

example Let $L: R^4 \to R^3$ be a L.T.
Take $B = \{(0,4,0,1), (-2,5,0,2), (-3,5,1,1), (-1,2,0,1)\}$ basis of R^4.
(you can verify that this a basis)

Let $L(0,4,0,1) = (3,1,2)$
$L(-2,5,0,2) = (2,-1,1)$
$L(-3,5,1,1) = (-4,3,0)$
& $L(-1,2,0,1) = (6,1,-1)$
i.e. values of L are given for the basis elements B.
We will see that we can find value of L for any element of R^4
~~Let v = (-8,21,1,5)~~
Let $v = (-8,21,1,5)$ an elt of R^4

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we want to find $L(v)$.

Note that $[v]_B = (-1, 2, 1, 1)$

i.e $(-8, 21, 1, 5) = -1.(0,4,0,1) + 2(-2,5,0,2) + 1.(-3,5,1,1) + 1.(-1,2,0,1)$

$\Rightarrow L(v) = L(-1(0,4,0,1) + 2(-2,5,0,2) + (-3,5,1,1) + (-1,2,0,1))$

$= -1L(0,4,0,1) + 2L(-2,5,0,2) + L(-3,5,1,1) + L(-1,2,0,1)$ (because L is a L.T.)

$= -1(3,1,2) + 2(2,-1,1) + (-4,3,0) + (6,1,-1)$

$= (3,1,-1)$

So $L(-8, 21, 1, 5) = (3, 1, -1)$

So we can find $L(v)$ forall $v \in R^4$ by writting v as l.c. of elts of basis B & then using that L is a L.T.

i.e. if $v = k_1(0,4,0,1) + k_2(-2,5,0,2) + k_3(-3,5,1,1) + k_4(-1,2,0,1)$

then $L(v) = k_1(3,1,2) + k_2(2,-1,1) + k_3(-4,3,0) + k_4(6,1,-1)$

$= (3k_1 + 2k_2 - 4k_3 + 6k_4,\ k_1 - k_2 + 3k_3 + k_4,\ 2k_1 + k_2 - k_4)$

matrix of a L.T.

Let $L : V \to W$ be a L.T. and $\dim V = n$, $\dim W = m$.
Let $B = \{\beta_1, \beta_2 \ldots \beta_n\}$ & $C = \{\gamma_1, \gamma_2 \ldots \gamma_m\}$ are basis for v.s. V & W respectively.

clearly $L(\beta_i) \in W$. and C is basis of W. So $L(\beta_i)$ can be written as linear combination of elts of C.

Let
$$L(\beta_1) = a_{11}\gamma_1 + a_{21}\gamma_2 + \cdots + a_{m1}\gamma_m$$
$$L(\beta_2) = a_{12}\gamma_1 + a_{22}\gamma_2 + \cdots + a_{m2}\gamma_m$$
$$\vdots$$
$$L(\beta_n) = a_{1n}\gamma_1 + a_{2n}\gamma_2 + \cdots + a_{mn}\gamma_m$$

then
$$A_{BC} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix}$$

is called matrix of L w.r.t. basis B & C of V & W.

Sometimes it is also written as $[L]_B^C$ or A_{BC}

i.e. coeff. of $L(\beta_1)$ as linear combination of elts of C forms first column of A
lly coefficients of $L(\beta_2)$ as l.c. of elts of C forms second column of matrix A

... and so on..

Example $L: R^4 \to R^3$ is a L.T. and
$B = \{(0,4,0,1), (-2,5,0,2), (-3,5,1,1)$
$(-1,2,0,1)\}$ is basis for R^4
and $C = \{(1,0,0), (0,1,0), (0,0,1)\}$ is basis for R^4

Let $L(0,4,0,1) = (3,1,2)$
$L(-2,5,0,2) = (2,-1,1)$
$L(-3,5,1,1) = (-4,3,0)$
$L(-1,2,0,1) = (6,1,-1)$

Then $L(0,4,0,1) = (3,1,2) = 3(1,0,0)+1(0,1,0)+2(0,0,1)$
$L(-2,5,0,2) = (2,-1,1) = 2(1,0,0)-1(0,1,0)+1(0,0,1)$
$L(-3,5,1,1) = (-4,3,0) = -4(1,0,0)+3(0,1,0)+0(0,0,1)$
B $L(-1,2,0,1) = (6,1,-1) = 6(1,0,0)+1(0,1,0)-1(0,0,1)$

So $[L]_B^C = A_{BC} = \begin{pmatrix} 3 & 2 & -4 & 6 \\ 1 & -1 & 3 & 1 \\ 2 & 1 & 0 & -1 \end{pmatrix}$

is matrix of L w.r.t. basis B & C.

If we change B or C the matrix of L will change.

So matrix of a L.T. depends on the basis.

~~note that in this example~~
note that $[L(0,4,0,1)]_C = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

$[L(-2,5,0,2)]_C = \begin{pmatrix} 2 \\ -1 \\ \cdot \end{pmatrix}$

& so on.. because C is standard ordered basis of R^3.

(Always remember $[v]_B \to$ coordinate vector of v w.r.t. basis B:
~~i.e column vector when we~~
i.e when we write v as l.c. of elements of basis B then the coefficients give $[v]_B$).
Recall $[v]_B$

Example Let $L: P_3 \to R^3$ given by
$L(a_3x^3 + a_2x^2 + a_1x + a_0) = (a_0 + a_1, 2a_2, a_3 - a_0)$

consider the standard ordered basis $B = \{x^3, x^2, x, 1\}$ for P_3 and $C = \{e_1, e_2, e_3\}$ for R^3.

clearly $L(x^3), L(x^2), L(x), L(1)$ will be in R^3 ~~write~~ so we can write the linear combination of elts of basis C. i.e

$L(x^3) = (0,0,1) = 0e_1 + 0e_2 + 1e_3$
$L(x^2) = (0,2,0) = 0e_1 + 2e_2 + 0e_3$
$L(x) = (1,0,0) = 1e_1 + 0e_2 + 0e_3$
$L(1) = (1,0,-1) = 1e_1 + 0e_2 - 1e_3$

So

$$A_{BC} = \begin{array}{c} [L(x^3)]_C \; [L(x^2)]_C \; [L(x)]_C \; [L(1)]_C \\ \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

Note that matrix A is 3×4

~~we will compute $L(5x^3 - x^2 + 3x + 2)$ from this matrix.~~

~~note $[(5x^3 - x^2 + 3x + 2)]_B = (5,-1,3,2)$~~

~~So $[L(5x^3 - x^2 + 3x + 2)]_C$~~

~~Thm 5.5 Let $L: V \to W$ is a L.T.~~

i.e if $L: V \to W$ is a L.T. and $\dim V = n$, $\dim W = m$ then the matrix of L will be $m \times n$ matrix.

Thm 5.5. Let $L: V \to W$ be a L.T. $\dim V = n$ & $\dim W = m$.
Let B & C are basis of V & W & A_{BC} is matrix of L w.r.t. B & C
Then for any vector $v \in V$

$$A_{BC}[v]_B = [L(v)]_C$$

So in above example

$$[L(5x^3-x^2+3x+2)]_C = A_{BC}[5x^3-x^2+3x+2]_B$$

$$= \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \\ 3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$$

$\cancel{\Rightarrow L(5x^3-x^2+3x+2) = 5x^2-2x+3}$

$\Rightarrow L(5x^3-x^2+3x+2) = (5,-2,3)$

See Table 5.1. matrices are given for various linear Transformations

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There is a topic "Transition matrix" which we should have done before starting the chapter Matrix of linear Transformation. So we will do Transition matrix first and then will proceed further as it is used in the coming topics.

I will explain Transition matrix by an example.

Let $V = R^2$

& $S = \{(1,1), (1,0)\}$ &
$T = \{(1,2), (1,-1)\}$ are two bases of R^2

Now express vectrs of S as linear combination of elts of T.

Clearly $(1,1) = \frac{3}{2}(1,2) - \frac{1}{2}(1,-1)$

& $(1,0) = \frac{1}{3}(1,2) + \frac{2}{3}(1,-1)$

$\Rightarrow$ So $P_{T \leftarrow S} = \begin{pmatrix} \frac{3}{2} & \frac{1}{3} \\ -\frac{1}{2} & \frac{2}{3} \end{pmatrix}$

is called transition matrix from basis S to basis T.

Ily if we write elts of T as linear combination of elts of S the coefficients will give transition matrix from T to S

i.e

~~$(1,2) = 1(1,1) + 2(1,0)$~~

i.e $(1,2) = 2(1,1) - 1(1,0)$

& $(1,-1) = -1(1,1) + 2(1,0)$

$\Rightarrow$ $P_{S \leftarrow T} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$P_{S \leftarrow T}$ is called transition matrix from basis T to basis S.

Note that $P_{S \leftarrow T} = P_{T \leftarrow S}^{-1}$

i.e $P_{S \leftarrow T}$ & $P_{T \leftarrow S}$ are inverse of each other.

sometimes they use symbols $P_{S\leftarrow T}$ and $Q_{T\leftarrow S}$ instead of $P_{S\leftarrow T}$ and $P_{T\leftarrow S}$.

Also one more result.

$$[v]_S = P_{S\leftarrow T}[v]_T \quad \forall\, v \in V$$

where $[v]_S$ & $[v]_T$ are coordinate vectors of v with respect to basis S and T respectively.

example let $V = \mathbb{R}^2$

$S = \{(1,1), (1,0)\}$ & $T = \{(1,2), (1,-1)\}$

as in earlier example.

so $P_{S\leftarrow T} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

Now let $v = (1,3)$. find $[v]_T$

so write v as linear combination of T

$$(1,3) = \frac{4}{3}(1,2) - \frac{1}{3}(1,-1).$$

so $[v]_T = \begin{pmatrix} \frac{4}{3} \\ -\frac{1}{3} \end{pmatrix}$

so $[v]_S = P_{S\leftarrow T}[v]_T$

$$= \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} \frac{4}{3} \\ -\frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{8}{3} + \frac{1}{3} \\ -\frac{4}{3} - \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

you can check this as

$(1,3) = 3(1,1) - 2(1,0)$

so $[(1,3)]_S = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

—

Now we come back to Linear Transformation matrix representation

example Let $L : P_3 \to R^3$ s.t.

$L(a_3x^3 + a_2x^2 + a_1x + a_0) = (a_0 + a_1, 2a_2, a_3 - a_0)$

(same L as in previous example)

$D = \{x^3 + x^2, x^2 + x, x + 1, 1\}$ basis of P_3

$E = \{(-2, 1, -3), (1, -3, 0), (3, -6, 2)\}$ basis of R^3

Find matrix of L w.r.t. D & E basis

Now $L(x^3 + x^2) = (0, 2, 1)$

$L(x^2 + x) = (1, 2, 0)$

$L(x + 1) = (2, 0, -1)$

$L(1) = (1, 0, -1)$

next write each of these images as linear combination of basis E of R^3.
For this we will make use of transition matrix $Q_{E \leftarrow C}$ where C is standard ordered basis of R^3

So $Q_{E\leftarrow C} = P_{C\leftarrow E}^{-1}$

$$= \begin{pmatrix} -2 & 1 & 3 \\ 1 & -3 & -6 \\ -3 & 0 & 2 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -6 & -2 & 3 \\ 16 & 5 & -9 \\ -9 & -3 & 5 \end{pmatrix}$$

So

$$[L(x^3+x^2)]_E = Q[L(x^3+x^2)]_C$$
$$= Q\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

$$[L(x^2+x)]_E = Q[L(x^2+x)]_C$$
$$= Q\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} -10 \\ 26 \\ -15 \end{pmatrix}$$

$$[L(x+1)]_E = Q[L(x+1)]_C$$
$$= Q\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$
$$= \begin{pmatrix} -15 \\ 41 \\ -23 \end{pmatrix}$$

&

$$[L(1)]_E = Q\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -9 \\ 25 \\ -14 \end{pmatrix}$$

So matrix of L w.r.t. basis D & E is

$$A_{DE} = \begin{bmatrix} -1 & -10 & -15 & -9 \\ 1 & 26 & 41 & 25 \\ -1 & -15 & -23 & -14 \end{bmatrix}$$

~~Next we will $L(5x^3 - x^2 + 3x + 2)$~~

Find the new matrix of a L.T. after change of basis

Let $L: V \to W$ be a L.T. & V & W are f.d.v.s.
Let B & C are basis of V & W respectively. So we can find matrix of L w.r.t. B & C i.e A_{BC}

Now let D & E are another basis of V & W respectively. so we can find matrix of L w.r.t. D & E also i.e A_{DE}.

Theorem 5.6 gives realation between these two matrices of L i.e A_{BC} & A_{DE}.
Thm 5.6 says that

$$A_{DE} = Q A_{BC} P^{-1}$$

where P is transition matrix from B to D & Q is transition matrix from C to E.

(Read Thm 5.6 statement)

Do example 5 yourself. It verifies Thm 5.6.

<u>linear operators & Similarity</u>:

- Let L is a linear operator from V to V. i.e. $L: V \to V$ is a L.T.
  Let B is a basis of V so we can find A_{BB}.
  Let C is another basis of V so we can find A_{CC}

By Theorem 5.6

$$A_{BB} = P^{-1} A_{CC} P$$

where P is transition matrix from B to C.
So A_{BB} & A_{CC} are Similar matrices.

i.e Two matrices of a L.O. with respect to different bases are Similar matrices.

Matrix for the Composition of Linear transformations

Let V_1, V_2, V_3 are f.d.v.s.

& $L_1 : V_1 \to V_2$ & $L_2 : V_2 \to V_3$ are ~~ar~~ are linear transformations.
Then $L_2 \circ L_1 : V_1 \to V_3$ is composition of L_1 & L_2.

Let B, C, D are bases of V_1, V_2, V_3.

Thm 5.7 says that matrix of $L_2 \circ L_1$ w.r.t. B & D i.e. A_{BD} is product of $A_{CD} A_{BC}$ i.e. $A_{BD} = A_{CD} A_{BC}$

where A_{CD} is matrix of L_2 ~~w.r.t.~~ w.r.t. bases C & D & A_{BC} is matrix of L_1 w.r.t. bases B & C.

See solved example 7.

For ~~Exercise~~ Exercise 5.2 do questions from 1 to 6. You can do upto 11.