ASP

2. Power Series.

Q1. Find radius of cgt & interval of cgt for the power series

$$\sum_{n=0}^{\infty} 2^{-n} x^{3n}$$

Pf

$$a_n = \begin{cases} 2^{-\frac{n}{3}} & \text{if } n=3m \text{ for } m=0,1,\dots \\ 0 & \text{otherwise} \end{cases}$$

$$|a_n|^{1/n} = \begin{cases} 2^{-\frac{1}{3}} & , \text{if } 3|n \\ 0 & , \text{Otherwise} \end{cases}$$

Thus radius of cgt of power series $\sum_{n=0}^{\infty} 2^{-n} x^{3n}$ is given by

$$\frac{1}{R} = \underset{n}{\text{lt sup}} |a_n|^{1/n} = 2^{-1/3}$$

$\therefore$ Radius of cgt is $R = 2^{1/3}$

& Interval of cgt is $(-2^{1/3}, 2^{1/3})$

Q No 2. Find radius of cgt & interval of cgt of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n \qquad \text{——— (1)}$$

Also discuss its interval of cgt

Pf: $a_n = \dfrac{(-1)^{n+1}}{n}$, Thus $|a_n|^{1/n} = \left|\dfrac{(-1)^{n+1}}{n}\right| = \dfrac{1}{n^{1/n}}$

radius of cgt R is given by

$$\frac{1}{R} = \underset{n}{\text{lt sup}} |a_n|^{1/n} = \cancel{\text{lt} \frac{1}{n^{1/n}}} = \underset{n}{\text{lt sup}} \frac{1}{n^{1/n}} = 1.$$

Thus $R=1$,

Interval of cgt is given by

$$|x-1|<1$$

i.e. $-1<x-1<1$

i.e. $0<x<2$

Thus Interval of cgt is $(0, 2)$

By Cauchy Hadmand Th, the power series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(x-1)^n$ is absolutely cgt for $|x-1|<1$

i.e. for $0<x<2$

The power series is dgt for $|x-1|>1$

Cgt for $R=1$

For $R=1$, the power series becomes

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$

(This is obtained by putting $x-1=1$ in (1))

This ~~series~~ alternating series is cgt by Leibnitz test

∴ ~~Series~~

Cgt for $R=-1$

For $R=-1$, the series (1) can be written as

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(-1)^n$$

[This obtained by putting $x-1=-1$ in (1)]

$$= -\sum_{n=1}^{\infty}\frac{1}{n}$$

This series is dgt.

Hence the given power series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$ is cgt for $0 < x \leq 2$ & dgt for $x < 0$ & $x > 2$

Q3 Find radius of cgt & interval of cgt for the power series

(i) $\sum_{n=0}^{\infty} n^2 x^n$

(ii) $\sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^n$

(iii) $\sum_{n=1}^{\infty} \frac{2^n}{n^2} x^n$

(iv) $\sum_{n=0}^{\infty} \left(\frac{n^3}{3^n}\right) x^n$

(v) $\sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$

(vi) $\sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \frac{1}{2^n} x^n$

(vii) $\sum_{n=1}^{\infty} \frac{3^n}{n 4^n} x^n$

(viii) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 4^n} x^n$

(IX) $\sum_{n=0}^{\infty} \frac{3^n}{\sqrt{n}} x^{2n+1}$

(X) $\sum \sqrt{n}\, x^n$

Proof (i) $\sum n^2 x^n$ (4)

$a_n = n^2$

$|a_n|^{1/n} = |n^2|^{1/n} = (n^{1/n})^2$

Since $\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} (n^{1/n})^2 = 1^2 = 1$,

$\lim\sup |a_n|^{1/n} = \lim_{n\to\infty} |a_n|^{1/n} = 1$

So radius of cgt R is given by

$$\frac{1}{R} = \lim\sup |a_n|^{1/n} = 1$$

$$\Rightarrow R = 1$$

Thus radius of cgt is 1 & interval of cgt is the open interval $(-1, 1)$

(ii) $$\sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n^n} x^n$$

$a_n = \frac{1}{n^n}$

$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n^n}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{n} = 0$

Hence $\lim_n \sup |a_n|^{1/n} = 0$

Thus radius of cgt $R = \infty$

& interval of cgt is $(-\infty, \infty)$

(iii) $\sum \frac{2^n}{n^2} x^n$

$a_n = \frac{2^n}{n^2}$

$|a_n|^{1/n} = \left[\frac{2^n}{n^2}\right]^{1/n} = \frac{2}{(n^{1/n})^2}$

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$\therefore \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{2}{(n^{1/n})^2} = \frac{2}{1^2} = 2$

$\therefore \lim_n \sup |a_n|^{1/n} = \lim_{n\to\infty} |a_n|^{1/n} = 2$

Hence radius of cgt $R = \frac{1}{\lim_n \sup |a_n|^{1/n}} = \frac{1}{2}$

& interval of cgt is open interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$

(iv) $\sum \frac{n^3}{3^n} x^n$

$a_n = \frac{n^3}{3^n}$.

$|a_n|^{1/n} = \left(\frac{n^3}{3^n}\right)^{1/n} = \frac{(n^{1/n})^3}{3}$.

$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{(n^{1/n})^3}{3} = \frac{1}{3}$

Thus $\lim_n \sup |a_n|^{1/n} = \lim_{n\to\infty} |a_n|^{1/n} = \frac{1}{3}$

Hence radius of cgt R is given by

$$\frac{1}{R} = \lim_n \sup |a_n|^{1/n} = \frac{1}{3}$$

Hence $R = 3$

Interval of cgt is ~~the~~ open interval $(-3, 3)$.

(v) $\sum \frac{2^n}{n!} x^n$

$a_n = \frac{2^n}{n!}$, $a_{n+1} = \frac{2^{n+1}}{(n+1)!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}\right| = \frac{2^{n+1} \cdot n!}{(n+1)!\ 2^n} = \frac{2}{(n+1)}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2}{(n+1)} = 0$$

$\therefore$ Radius of cgt R is given by

$$\cancel{\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0}$$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \infty$$

Hence radius of cgt ∞ & interval of cgt is $(-\infty, \infty)$.

(vi) $\sum \frac{1}{(n+1)^2\, 2^n} x^n$

$$a_n = \frac{1}{(n+1)^2\, 2^n}, \quad a_{n+1} = \frac{1}{(n+2)^2}\frac{1}{2^{n+1}}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{(n+2)^2\, 2^{n+1}} \cdot (n+1)^2\, 2^n = \left(\frac{n+1}{n+2}\right)^2 \cdot \frac{1}{2}$$

$$= \left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)^2 \frac{1}{2}$$

Thus $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right]^2 \cdot \frac{1}{2} = \frac{1}{2}$

$\therefore$ Radius of cgt $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = 2$

Interval cgt is $(-2, 2)$.

(vii) $\sum \frac{3^n}{n\,4^n} x^n$

$$a_n = \frac{3^n}{n\,4^n}$$

$$\underset{n}{\text{lt}} \sup |a_n|^{1/n} = \underset{n}{\text{lt}} \sup \left(\frac{3^n}{n\,4^n}\right)^{1/n} = \underset{n}{\text{lt}} \sup \left(\frac{3}{n^{1/n}\,4}\right)$$

$$= \frac{3}{4}\,\underset{n}{\text{lt}} \sup \left(\frac{1}{n^{1/n}}\right) = \frac{3}{4} \quad \left(\text{Since } \underset{n\to\infty}{\text{lt}} n^{1/n} = 1\right)$$

$\therefore$ radius of cgt R of the power series

$$\sum \frac{3^n}{n\,4^n} x^n$$

is given by

$$\frac{1}{R} = \underset{n}{\text{lt}} \sup |a_n|^{1/n} = \frac{3}{4}$$

$\therefore$ radius of cgt is $R = \frac{4}{3}$

& interval of cgt is $\left(-\frac{4}{3}, \frac{4}{3}\right)$.

(viii) $\sum \frac{(-1)^n}{n^2\,4^n} x^n$

$$a_n = \frac{(-1)^n}{n^2\,4^n}$$

$$\therefore |a_n|^{1/n} = \left|\frac{(-1)^n}{n^2\,4^n}\right|^{1/n} = \frac{1}{(n^{1/n})^2\,4}$$

$$\therefore \underset{n\to\infty}{\text{lt}} |a_n|^{1/n} = \underset{n\to\infty}{\text{lt}} \frac{1}{(n^{1/n})^2 \cdot 4} = \frac{1}{4}$$

Hence radius of cgt R is given by

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \frac{1}{4}$$

∴ radius of cgt of the power series is
$R = 4$ & interval of cgt is $(-4, 4)$

(ix) $\sum_{n=10}^{\infty} \frac{3^n}{\sqrt{n}} x^{2n+1}$ $\qquad m = 2n-1$, $\frac{m-1}{2}$

$$a_n = \begin{cases} \dfrac{3^{\frac{n-1}{2}}}{\sqrt{\frac{n-1}{2}}}, & n = 1, 3, 5, \ldots \text{ (odd)} \\ 0, & n = 0, 2, 4, \ldots \text{ (even)} \end{cases}$$

$$|a_n|^{1/n} = \begin{cases} \dfrac{\left[3^{\frac{n-1}{2}}\right]^{\frac{1}{n}}}{\left(\frac{n-1}{2}\right)^{1/n}}, & n = 1, 3, 5, \ldots \\ 0, & n = 0, 2, 4, \ldots \end{cases}$$

$$= \begin{cases} \dfrac{3^{1/2} \cdot \left(3^{\frac{1}{n}}\right)^{-\frac{1}{2}}}{\frac{(n-1)^{1/n}}{2^{1/n}}}, & n = 1, 3, 5, \ldots \\ 0, & n = 0, 2, 4, \ldots \end{cases}$$

$$= \begin{cases} \dfrac{3^{1/2} \cdot \left(3^{1/n}\right)^{-\frac{1}{2}} 2^{1/n}}{(n-1)^{1/n}}, & n = 1, 3, 5, \ldots \\ 0, & n = 0, 2, 4, \ldots \end{cases}$$

Now

$$\lim_{n\to\infty} \frac{3^{1/2} \cdot (3^{1/n})^{-1/2}\, 2^{1/n}}{(n-1)^{1/n}} = 3^{1/2}$$

Thus

$$\lim_{n} \sup |a_n|^{1/n} = 3^{1/2} = \sqrt{3}$$

Hence radius of cgt R is given by

$$\frac{1}{R} = \lim_{n} \sup |a_n|^{1/n} = \sqrt{3}$$

i.e radius of cgt is $R = \frac{1}{\sqrt{3}}$

& interval of cgt is open interval $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

(8) $\sum \sqrt{n}\, x^n$

$a_n = \sqrt{n}$

Radius of cgt R is given by

$$\frac{1}{R} = \lim_{n} \sup |a_n|^{1/n} = \lim_{n} \sup (\sqrt{n})^{1/n}$$

$$= \lim_{n} \sup (n^{1/n})^{1/2} = 1$$

$\therefore$ radius of cgt of power series $\sum \sqrt{n}\, x^n$ is $R = 1$ & interval of cgt is open interval $(-1, 1)$.

Theorem Let R be the radius of cgt of $\sum a_n x^n$ & let K be a closed and bounded interval contained in the interval of cgt $(-R, R)$. Then the power series cgs uniformly on K.

Pf: $K \subseteq (-R, R)$. Let $x \in K$. Then $|x| < R$

Thus For any $x \in K$, $0 < \frac{|x|}{R} < 1$

Choose a positive real number c such that

$$\frac{|x|}{R} < c < 1$$

Thus, $|x| < cR$, $\forall x \in K$

Now

$$\limsup_n |a_n x^n|^{1/n} = \limsup_n |x| |a_n|^{1/n}$$

$$= |x| \limsup_n |a_n|^{1/n}$$

$$= \frac{|x|}{R} < c$$

Thus $\exists$ a positive integer N s.t

$$|a_n x^n|^{1/n} < c \quad \forall n \geq N$$

Hence ~~for~~ for every $x \in K$, $|a_n x^n| < c^n \quad \forall n \geq N$

Since $0 < c < 1$, the geometric series $\sum c^n$ is cgt.

By M-test

$\sum a_n x^n$ is uniform cgt on K.

Theorem: A power series $\sum_{n=0}^{\infty} a_n x^n$ and a series obtained by term by term differentiation,

$$\sum_{n=1}^{\infty} n a_n x^{n-1}$$

have same radius of cgt.

Suppose R is radius of cgt of the series $\sum_{n=0}^{\infty} a_n x^n$

Then $\frac{1}{R} = \underset{n}{\text{lt}} \sup |a_n|^{1/n}$

Now $\underset{n\to\infty}{\text{lt}} n^{1/n} = 1$. Thus

$$\underset{n}{\text{lt}} \sup |n a_n|^{1/n} = \underset{n}{\text{lt}} \sup \left(n^{1/n} |a_n|^{1/n}\right)$$

$$= \underset{n}{\text{lt}} \sup |a_n|^{1/n} = \frac{1}{R}$$

Thus R is radius of cgt of the power series $\sum_{n=1}^{\infty} n a_n x^{n-1}$

Note: Since a power series is uniform cgt on each closed and bounded interval contained in the inverval of cgt of the power series and power series obtained by ~~term~~ term-by-term differentiative have same radius of cgt, the power series can be differentiated term-by-term inside the interval of cgt. That is if R is radius of cgt of the power series $\sum a_n x^n$

If $f(x) = \sum_{n=1}^{\infty} a_n x^n$ for $|x| < R$

then $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ for $|x| < R$.

Theorem Power series $\sum_{n=0}^{\infty} a_n x^n$ & $\sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$ have same radius of cgt

Let R be radius of cgt of $\sum a_n x^n$, Then

$$\frac{1}{R} = \lim_n \sup |a_n|^{1/n}$$

Now $\lim_n \sup \left|\frac{a_n}{n+1}\right|^{1/n} = \lim_n \sup \frac{1}{(n+1)^{1/n}} |a_n|^{1/n}$

$= \lim \sup |a_n|^{1/n}$ $\left(\text{Since } \lim_{n\to\infty} (n+1)^{1/n} = 1\right)$

$= \frac{1}{R}$

Thus both the power series & series obtained by term by term integration have same radius of cgt.