

Riemann Integration

BSc (Hons) IV Sem Real Analysis ASP

(1)

Def let f be a bounded function defined on a closed interval $[a, b]$. By a partition of $[a, b]$, we mean a set of points

$\{x_0, x_1, \dots, x_n\}$ s.t. $a = x_0 < x_1 < \dots < x_n = b$
we write $P: a = x_0 < x_1 < \dots < x_n = b$.
We write $\Delta x = x_i - x_{i-1}$.

$$M_i = \sup \{f(x) \mid x_{i-1} \leq x \leq x_i\}$$

$$m_i = \inf \{f(x) \mid x_{i-1} \leq x \leq x_i\}$$

$$M = \sup_{x \in [a, b]} f(x) \quad \text{and} \quad m = \inf_{x \in [a, b]} f(x)$$

Upper Darboux sum $U(f, P)$ of f is defined by \wedge corresponding to a partition P

$$U(f, P) = \sum_{i=1}^n M_i \Delta x_i$$

and Lower Darboux sum $L(f, P)$ of f corresponding to a partition P is defined by

$$L(f, P) = \sum_{i=1}^n m_i \Delta x_i$$

We define upper Darboux Integral \mathcal{U} of f over $[a, b]$ by

$$\mathcal{U}(f) = \inf_P U(f, P)$$

and lower Darboux Integral of f by

$$\mathcal{L}(f) = \sup_P L(f, P)$$

Upper Darboux Integral of f over $[a, b]$ is also denoted by $\int_a^b f(x) dx$ & lower Darboux Integral is denoted by $\int_a^b f(x) dx$

f is said to be Darboux integrable or simply integrable over $[a, b]$ if $U(f) = L(f)$ i.e. $\int_a^b f = \int_a^b f$

If f is integrable over $[a, b]$, then we write

$$\int_a^b f = \int_a^b f = \int_a^b f, \text{ i.e. } \int_a^b f = U(f) = L(f).$$

Definition

Let f be bounded function defined on a closed interval $[a, b]$. Then for any partition P of $[a, b]$, we have

$$m(b-a) \leq L(f, P) \leq U(f, P) \leq M(b-a), \text{ where}$$

$$M = \sup_{x \in [a, b]} f(x) \text{ \& } m = \inf_{x \in [a, b]} f(x).$$

Pf: Let $P: a = x_0 < x_1 < \dots < x_n = b$ be any partition of $[a, b]$

$$\text{let } M_i = \sup \{ f(x) \mid x_{i-1} \leq x \leq x_i \}$$

$$m_i = \inf \{ f(x) \mid x_{i-1} \leq x \leq x_i \}$$

$$M = \sup_{x \in [a, b]} f(x) \text{ \& } m = \inf_{x \in [a, b]} f(x).$$

clearly $m \leq m_i \leq M_i \leq M \quad \forall i = 1, 2, \dots, n$

$$\Rightarrow \sum_{i=1}^n m \Delta x_i \leq \sum_{i=1}^n m_i \Delta x_i \leq \sum_{i=1}^n M_i \Delta x_i \leq \sum_{i=1}^n M \Delta x_i$$

$$\Rightarrow m \sum_{i=1}^n \Delta x_i \leq L(f, P) \leq U(f, P) \leq M \sum_{i=1}^n \Delta x_i$$

$$\Rightarrow m(b-a) \leq L(f, P) \leq U(f, P) \leq M(b-a),$$

$$\left(\begin{aligned} \text{Since } \sum_{i=1}^n \Delta x_i &= \Delta x_n + \Delta x_{n-1} + \dots + \Delta x_1 \\ &= (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \dots + (x_1 - x_0) \\ &= x_n - x_0 = b - a \end{aligned} \right)$$

Riemann definition of Integrability

Let f be a bounded function defined on closed interval $[a, b]$,
let $P: a = x_0 < x_1 < \dots < x_n = b$ be a partition of $[a, b]$.

Then

$$S(f, P) = \sum_{i=1}^n f(t_i) (x_i - x_{i-1}), \text{ where } x_{i-1} \leq t_i \leq x_i$$

is called a Riemann Sum of f corresponding to a partition P of $[a, b]$. There are infinitely many ~~so~~ Riemann ~~of~~ Sum of f corresponding to a given partition P (because $t_i \in [x_{i-1}, x_i]$).

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A bounded

function $f: [a, b] \rightarrow \mathbb{R}$ is said to be Riemann integrable
(written $f \in \mathcal{R}[a, b]$)
on $[a, b]$ if \exists a real number A s.t. for every

$\epsilon > 0 \exists \delta > 0$ s.t.

$$\mathcal{U}(P) < \delta \Rightarrow |S(f, P) - A| < \epsilon.$$

A is called Riemann integral of f over $[a, b]$
if we write

$$A = \int_a^b f = \int_a^b f(x) dx$$

Theorem Let f be a bounded function defined on closed
interval $[a, b]$. Then f is Riemann integrable if and only if
 f is Darboux Integrable on $[a, b]$

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Example Every constant function on $[a, b]$ is Riemann integrable on $[a, b]$,

Pf: Let $f(x) = k, x \in [a, b]$

Let $P: a = x_0 < x_1 < \dots < x_n = b$ be any partition of $[a, b]$

Then $S(f, P) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1})$ $\left(\begin{array}{l} \text{---} \\ t_i \in [x_{i-1}, x_i] \end{array} \right)$

$$= \sum_{i=1}^n k(x_i - x_{i-1})$$

$$= k \sum_{i=1}^n (x_i - x_{i-1})$$

$$= k[(x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \dots + (x_1 - x_0)]$$

$$= k[x_n - x_0]$$

$$= k(b-a)$$

Let $\epsilon > 0$ be given. Take $\delta = 1$,
if $\mu(P) < \delta$, then $|S(f, P) - k(b-a)| = 0 < \epsilon$

Hence f is Riemann integrable on $[a, b]$

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Def Let P be a partition of $[a, b]$. A partition θ of $[a, b]$ is called refinement of P if $P \subseteq \theta$, i.e. if $x_i \in P$, then $x_i \in \theta$.

Lemma 1 Let f be a bounded function defined on closed interval $[a, b]$. If P and θ are two partitions of $[a, b]$ s.t. θ is refinement of P , then

$$L(f, P) \leq L(f, \theta) \leq U(f, \theta) \leq U(f, P)$$

Lemma 2 Let f be a bounded function defined on closed interval $[a, b]$. If P and θ are two partitions of $[a, b]$, then

$$L(f, P) \leq U(f, \theta)$$

Pf:

Since $P \subseteq P \cup \theta$,

$$L(f, P) \leq L(f, P \cup \theta) \quad \text{--- (1)}$$

Since $\theta \subseteq P \cup \theta$, we have

$$U(f, P \cup \theta) \leq U(f, \theta) \quad \text{--- (2)}$$

Also

$$L(f, P \cup \theta) \leq U(f, P \cup \theta) \quad \text{--- (3)}$$

Thus

$$L(f, P) \leq L(f, P \cup \theta) \leq U(f, P \cup \theta) \leq U(f, \theta)$$

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Theorem If f is a bounded function on $[a, b]$, then
$$L(f) \leq U(f).$$

Proof Let P be any partition of $[a, b]$. Then

$$L(f, P) \leq U(f, \theta) \quad \text{for all partition } \theta \text{ of } [a, b].$$

~~Then~~
ie.
$$L(f, P) \leq \{U(f, \theta) \mid \theta \text{ is a partition of } [a, b]\}.$$

$$\Rightarrow L(f, P) \leq \inf_{\theta} U(f, \theta) = U(f)$$

ie. $L(f, P) \leq U(f)$ for every partition P of $[a, b]$

Hence
$$\sup_P L(f, P) \leq U(f)$$

ie.
$$\boxed{L(f) \leq U(f)}$$

Theorem Let f be a bounded function on closed interval $[a, b]$.
~~Then~~ Then f is integrable if and only if for each $\epsilon > 0$
 \exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon$$

If Suppose that f is integrable, let $\epsilon > 0$. Then \exists partitions

$$P_1 \text{ \& } P_2 \text{ of } [a, b] \text{ s.t.} \quad \text{Since } L(f) = \sup L(f, P)$$

$$\text{Since } L(f) = \sup_p L(f, P), \exists \text{ a partition } P_1 \text{ s.t.} \quad L(f)$$

$$P_1 \text{ s.t.} \quad L(f) - \frac{\epsilon}{2} < L(f, P_1) \quad \text{--- (1) } \overline{U(f)} \overline{U(f, P)}$$

$$\text{Since } U(f) = \inf_p U(f, P), \exists \text{ a partition } P_2 \text{ s.t.}$$

$$U(f, P_2) < U(f) + \frac{\epsilon}{2} \quad \text{--- (2)}$$

~~(1) & (2)~~ Take $P = P_1 \cup P_2$.

$$(1) \Rightarrow L(f) - \frac{\epsilon}{2} < L(f, P_1) \leq L(f, P_1 \cup P_2) = L(f, P) \quad \text{--- (3)}$$

$$(2) \Rightarrow U(f, P) = U(f, P_1 \cup P_2) \leq U(f, P_2) < U(f) + \frac{\epsilon}{2} \quad \text{--- (4)}$$

(1) ~~(2)~~, (3), (4) \Rightarrow

$$\begin{aligned} U(f, P) - L(f, P) &< U(f, P) + \frac{\epsilon}{2} - \left(L(f, P) - \frac{\epsilon}{2} \right) \\ &= U(f) - L(f) + \epsilon \end{aligned}$$

Since f is integrable $U(f) = L(f)$

Then

$$U(f, P) - L(f, P) < \epsilon$$

Conversely suppose that for $\epsilon > 0 \exists$ a partition P

s.t.

$$U(f, P) - L(f, P) < \epsilon \quad (5)$$

$$\text{Now } U(f) \leq U(f, P) = U(f, P) - L(f, P) + L(f, P)$$

$$< \epsilon + L(f, P) \quad \text{by (5)}$$

$$\leq \epsilon + L(f)$$

Since ϵ is arbitrary,

$$\cancel{L(f)} \quad U(f) \leq L(f) \quad (6)$$

Also $L(f) \leq U(f) \implies$

Combining (6) & (7), we get

$$U(f) = L(f)$$

Thus f is R integrable.

Example Let $[a, b]$ be an interval. Let $f(x)$ be a function defined on a closed interval $[a, b]$ by

$$f(x) = \begin{cases} 1 & \text{for } x \text{ rational in } [a, b] \\ 0 & \text{otherwise} \end{cases}$$

Show that f is not integrable.

Pf: Let $P: a = x_0 < x_1 < \dots < x_n = b$.

be a partition of $[a, b]$

$$M_i = \sup \{ f(x) \mid x \in [x_{i-1}, x_i] \}$$

$$m_i = \inf \{ f(x) \mid x \in [x_{i-1}, x_i] \}$$

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Thm

$$U(f, P) = \sum_{i=1}^n M_i \Delta x_i = \sum_{i=1}^n \Delta x_i = \cancel{b-a}$$
$$= (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) - \dots + (x_1 - x_0)$$
$$= x_n - x_0$$
$$= b - a$$

$$L(f, P) = \sum_{i=1}^n m_i \Delta x_i = 0$$

$$\therefore U(f) = \inf_P U(f, P) = 1$$

$$\& L(f) = \sup_P L(f, P) = 0$$

$$\text{Thm } U(f) \neq L(f)$$

Hence f is not \mathbb{R} integrable on $[a, b]$.