Riemann Integration

Def Let f be a bounded function defined on a closed interval $[a,b]$. By a partition of $[a,b]$, we mean a set of points $\{x_0, x_1, \ldots x_n\}$ s.t. $a = x_0 < x_1 < \ldots < x_n = b$
& we write $P : a = x_0 < x_1 < \ldots < x_n = b$.
We write $\Delta x = x_i - x_{i-1}$.

$$M_i = \sup\{f(x) \mid x_{i-1} \le x \le x_i\}$$

$$m_i = \inf\{f(x) \mid x_{i-1} \le x \le x_i\}$$

$$M = \sup_{x \in [a,b]} f(x) \quad \text{and} \quad m = \inf_{x \in [a,b]} f(x).$$

Upper Darboux sum $U(f,P)$ of f corresponding to a partition P is defined by

$$U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i$$

and Lower Darboux Sum $L(f,P)$ of f corresponding to a partition P is defined by

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i$$

We define upper Darboux Integral of f over $[a,b]$ by

$$U(f) = \inf_P U(f,P)$$

and lower Darboux integral of f by

$$L(f) = \sup_P L(f,P)$$

Upper Darboux Integral of f over [a,b] is also denoted by $\overline{\int_a^b} f(x)dx$ & lower Darboux Integral is denoted by $\underline{\int_a^b} f(x)\,dx$

f is said to be Darboux integrable or simply integrable over [a,b] if $U(f) = L(f)$ i.e $\overline{\int_a^b} f = \underline{\int_a^b} f$

If f is integrable over [a,b], then we write

$$\int_a^b f = \overline{\int_a^b} f = \underline{\int_a^b} f, \quad \text{i.e } \int_a^b f = U(f) = L(f).$$

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Th. Let f be bounded function defined on a closed interval [a,b]. Then for any partition P of [a,b], we have

$$m(b-a) \leq L(f,P) \leq U(f,P) \leq M(b-a), \quad \text{where}$$

$M = \sup\limits_{x\in[a,b]} f(x)$ & $m = \inf\limits_{x\in[a,b]} f(x)$.

Pf: Let $P: a = x_0 < x_1 < \cdots < x_n = b$ be any partition of [a,b]

Let $M_i = \sup\{f(x) \mid x_{i-1} \leq x \leq x_i\}$

$m_i = \inf\{f(x) \mid x_{i-1} \leq x \leq x_i$

$M = \sup\limits_{x\in[a,b]} f(x)$ & $m = \inf\limits_{x\in[a,b]} f(x)$.

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Clearly $\quad m \leq m_i \leq M_i \leq M \qquad \forall\ i = 1, 2, \ldots n$

$$\Rightarrow \sum_{i=1}^{n} m\,\Delta x_i \leq \sum_{i=1}^{n} m_i\,\Delta x_i \leq \sum_{i=1}^{n} M_i \leq \sum_{i=1}^{n} M\,\Delta x_i$$

$$\Rightarrow m \sum_{i=1}^{n} \Delta x_i \leq L(f, P) \leq U(f, P) \leq m \sum_{i=1}^{n} \Delta x_i$$

$$\Rightarrow m(b-a) \leq L(f, P) \leq U(f, P) \leq m(b-a),$$

$$\left(\text{Since} \sum_{i=1}^{n} \Delta x_i = \Delta x_n + \Delta x_{n-1} + \cdots + \Delta x_1 \right.$$
$$= (x_n - \cancel{x_{n-1}}) + (\cancel{x_{n-1}} - x_{n-2}) + \cdots + (\cancel{x_1} - x_0)$$
$$\left. = x_n - x_0 = b - a \right)$$

Riemann definition of Integrability

Let f be a bounded function defined on closed interval $[a, b]$,
Let $P : a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a, b]$.
Then

$$S(f, P) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}), \text{ where } x_{i-1} \leq t_i \leq x_i$$

is called a Riemann Sum of f corresponding to a partition P of $[a,b]$. There are infinitely many ~~sa~~ Riemann ~~of~~ Sum of f corresponding to a given partition P (because $t_i \in [x_{i-1}, x_i]$).

A bounded function $f : [a,b] \longrightarrow \mathbb{R}$ is said to be Riemann integrable on $[a,b]$ (written $f \in R[a,b]$) if $\exists$ a real number A s.t. for every $\epsilon > 0$ $\exists$ $\delta > 0$ s.t.

$$\mu(P) < \delta \Rightarrow |S(f,P) - A| < \epsilon.$$

A is called Riemann integral of f over $[a,b]$ & we write

$$A = \int_a^b f = \int_a^b f(x)\,dx$$

<u>Theorem</u> Let f be a bounded function defined on closed interval $[a,b]$. Then f is Riemann integrable if and only if f is Darboux integrable on $[a,b]$

Example Every constant function on $[a,b]$ is Riemann integrable on $[a,b]$.

Pf. Let $f(x) = k$, $x \in [a,b]$

Let $P: a = x_0 < x_1 < \dots < x_n = b$ be any partition of $[a,b]$

Then
$$S(f,P) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \qquad (t_i \in [x_{i-1}, x_i])$$
$$= \sum_{i=1}^{n} k(x_i - x_{i-1})$$
$$= k\sum_{i=1}^{n} (x_i - x_{i-1})$$
$$= k[(x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \dots + (x_1 - x_0)]$$
$$= k[x_n - x_0)$$
$$= k(b-a)$$

Let $\epsilon > 0$ be given. Take $\delta = 1$,

if $\mu(P) < \delta$, then
$$|S(f,P) - k(b-a)| = 0 < \epsilon$$

Hence f is Riemann integrable on $[a,b]$

Def Let P be a partition of $[a,b]$. A partition Q of $[a,b]$ is called refinement of P if $P \subseteq Q$, i.e. if $x_i \in P$ then $x_i \in Q$.

Lemma 1 Let f be a bounded function defined on closed interval $[a,b]$. If P and Q are two partition of $[a,b]$ s.t Q is refinement of P, then

$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$$

Lemma 2 ~~If f is a~~ Let f be a bounded function defined on closed interval $[a,b]$. If P and Q are two partition of $[a,b]$, then

$$L(f,P) \leq U(f,Q)$$

Pf:

Since $P \subseteq P \cup Q$,

$$L(f,P) \leq L(f, P\cup Q) \quad \text{——— (1)}$$

Since $Q \subseteq P \cup Q$, we have

$$U(f, P\cup Q) \leq U(f,Q) \quad \text{——— (2)}$$

Also

$$L(f, P\cup Q) \leq U(f, P\cup Q) \quad \text{——— (3)}$$

Thus

$$L(f,P) \leq L(f, P\cup Q) \leq U(f, P\cup Q) \leq U(f,Q)$$

Theorem If f is a bounded function on $[a,b]$, then

$$L(f) \leq U(f).$$

Proof Let P be any partition of $[a,b]$. Then

$$L(f,P) \leq U(f,Q) \quad \text{for all partition } Q \text{ of } [a,b].$$

i.e. $L(f,P) \leq \{U(f,Q) \mid Q \text{ is a partition of } [a,b]\}$.

$$\Rightarrow L(f,P) \leq \inf_{Q} U(f,Q) = U(f)$$

i.e $L(f,P) \leq U(f)$ for every partition P of $[a,b]$

Hence $\sup_{P} L(f,P) \leq U(f)$

i.e $\boxed{L(f) \leq U(f)}$

Theorem Let f be a bounded function on closed interval $[a,b]$. ~~Then~~ Then f is integrable if and only if for each $\epsilon>0$ $\exists$ a partition P of $[a,b]$ such that

$$U(f,P)-L(f,P)<\epsilon$$

Pf Suppose that f is integrable. Let $\epsilon>0$. Then $\exists$ ~~partitions P_1 & P_2 of $[a,b]$ s.t.~~ ~~Since $L(f)=\sup_P L(f,P)$~~

Since $L(f)=\sup_P L(f,P)$, $\exists$ a partition P_1 s.t.

$$L(f)-\frac{\epsilon}{2}<L(f,P_1) \quad \text{——— (1)}$$

$L(f)$ ~~$U(f)$~~ $U(f,P)$

Since $U(f)=\inf_P U(f,P)$, $\exists$ a partition P_2 s.t.

$$U(f,P_2)<U(f)+\epsilon/2 \quad \text{——— (2)}$$

Take $P=P_1\cup P_2$.

$$(1)\Rightarrow \quad L(f)-\frac{\epsilon}{2}<L(f,P_1)\le L(f,P_1\cup P_2)=L(f,P) \quad \text{——— (3)}$$

$$(2)\Rightarrow U(f,P)=U(f,P_1\cup P_2)\le U(f,P_2)<U(f)+\epsilon/2 \quad \text{——— (4)}$$

(1), (2), (3), (4) $\Rightarrow$

$$U(f,P)-L(f,P) < U(f)+\epsilon/2-\left(L(f)-\epsilon/2\right)$$
$$= U(f)-L(f)+\epsilon$$

Since f is integrable $U(f)=L(f)$

Thus
$$U(f,P)-L(f,P) < \epsilon$$

Conversely suppose that for $\epsilon>0$ $\exists$ a partition P s.t.
$$U(f,P)-L(f,P) < \epsilon \quad \text{———} \quad (5)$$

Now
$$U(f) \leq U(f,P) = U(f,P)-L(f,P)+L(f,P)$$
$$< \epsilon + L(f,P) \qquad \text{by (5)}$$
$$\leq \epsilon + L(f)$$

Since ϵ is arbitrary,
$$\cancel{L(f)}\ U(f) \leq L(f) \quad \text{———} \quad (6)$$

Also $L(f) \leq U(f)$ — (7)

Combining (6) & (7), we get

$$U(f) = L(f)$$

Thus f is ~~R~~ integrable.

Example. ~~Let $[a,b]$ be an~~ Let $f(x)$ be a function defined on a closed interval $[a,b]$ by

$$f(x) = \begin{cases} 1 & \text{for } x \text{ rational in } [a,b] \\ 0 & \text{otherwise} \end{cases}$$

Show that f is not integrable.

Pf: Let $P: a = x_0 < x_1 < \dots < x_n = b$.

be a partition of $[a,b]$

$$M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$$
$$= 1$$
$$m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$$
$$= 0$$

Then $U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \Delta x_i = \cancel{b-a}$

$= (x_n - \cancel{x_{n-1}}) + (\cancel{x_{n-1}} - \cancel{x_{n-2}}) - - + (\cancel{x_1} - x_0)$

$= x_n - x_0$

$= b-a$

$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i = 0$

$\therefore \quad U(f) = \inf_P U(f, P) = 1$

$\& \quad L(f) = \sup_P L(f, P) = 0$

Thus $U(f) \neq L(f)$

Hence f is not ~~R~~ integrable on $[a, b]$.