INTEGRAL TRANSFORMS

FOURIER TRANSFORMS

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Fourier series is used to represent a periodic function by a discrete sum of complex exponentials (sines and cosines).

Fourier Transform is used to represent a non-periodic function by a continuous superposition or integral of complex exponentials.

Fourier Integral Theorem.

Assume that there is a function $f(x)$ which satisfies the following conditions

(i) $f(x)$ is piecewise continuous

(ii) $f(x)$ is differentiable

(iii) $f(x)$ is absolutely integrable i.e

$\int_{-\infty}^{\infty} |f(x)|dx$ converges in $(-\infty, \infty)$.

Then $$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos u(t-x)\,dt\,du$$

Proof: Let $f(x)$ be a periodic function with period $2L$ having Fourier series given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \quad -(1)$$

where $a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt$$

Substituting these values in eq. (1), we get,

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt + \sum_{n=1}^{\infty} \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\cos\frac{n\pi x}{L}\,dt$$
$$+ \sum_{n=1}^{\infty} \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\sin\frac{n\pi x}{L}\,dt$$

$$= \frac{1}{2L}\int_{-L}^{L} f(t)\,d(t) + \sum_{n=1}^{\infty}\frac{1}{L}\int_{-L}^{L} f(t)\left[\cos\frac{n\pi t}{L}\cos\frac{n\pi x}{L} + \sin\frac{n\pi t}{L}\sin\frac{n\pi x}{L}\right]dt$$

$$\therefore f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt + \sum_{n=1}^{\infty}\frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi}{L}(t-x)\,dt \quad \text{— (2)}$$

A non periodic function can be treated as a ~~b~~ periodic function whose period $2L \to \infty$.

In eq. (2), let $\frac{n\pi}{L} = u$, $\frac{\pi}{L} = \Delta u$

and $L \to \infty$

Then we have

$$f(x) \to \frac{1}{\pi}\sum_{n=1}^{\infty} \Delta u \int_{-\infty}^{\infty} f(t)\cos u(t-x)\,dt$$

$$\text{or, } f(x) = \frac{1}{\pi}\int_{0}^{\infty} du \int_{-\infty}^{\infty} f(t)\cos u(t-x)\,dt$$

On replacing the infinite sum by integration

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Fourier's Complex Integral

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\, du \int_{-\infty}^{\infty} f(t)\, e^{iut}\, dt$$

Fourier Transform

Fourier Transform $F(s)$ of a function $f(t)$ is given by

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$$

~~Invelverse~~

Inverse Fourier Transform

Inverse Fourier Transform $f(x)$ of a function $F(s)$ is given by

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

Fourier Cosine Transform ($F_c(s)$) of $f(x)$

$$\cancel{F_c(s) = F_c[f\ F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos S}$$

$$F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\, dx$$

Inverse Fourier Cosine Transform $f_c(x)$ of $F(s)$.

$$f_c(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \cos sx\, F(s)\, ds$$

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Fourier Sine Transform of $f(x)$

$$F_s(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

Inverse Fourier Sine Transform $f_s(x)$ of $F(s)$.

$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(s) \sin sx \, ds$$

Examples:

Ex.1. Find the Fourier Transform of

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

Solution:

The Fourier transform of a function $f(x)$ is given by

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx$$

Substituting the value of $f(x)$,

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 1 \cdot e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is} \right]_{-a}^{a} = \frac{1}{\sqrt{2\pi}} \frac{e^{ias} - e^{-ias}}{is}$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \frac{e^{ias} - e^{-ias}}{2i} \cdot \frac{1}{s}$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin sa}{s}$$

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Ex. 2. Find the Fourier Sine Transform of e^{-ax}.

Solution:

Fourier Sine transform of $f(x)$ is

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

Substituting the value of $f(x)$,

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx \quad \text{--- (1)}$$

$\because$ Using integration by parts,

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$$

Subs. in eq. (1),

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2}(-a \sin sx - s \cos sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{e^0(-s)}{a^2+s^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{s}{a^2+s^2}$$

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Ex. 3. Find the Fourier cosine transform of $f(x) = e^{-ax}$

Solution:

Fourier cosine transform of $f(x)$ is given by

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

Substituting the value of $f(x)$,

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx \quad \text{—} (1)$$

Using integration by parts,

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

Substituting this value in eq. (1),

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s\sin sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left(0 + \frac{a}{a^2+s^2}\right)$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$$

H.W.1. Show that Fourier transform of a Gaussian Function is Gaussian.