Lecture for week [16 March – 22 March; 2020]

Paper Name : Electricity and Magnetism

Course : B.Sc (Hons.) Physics

Semester : II

Section : A

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LECTURE FOR WEEK (16–22 MARCH 2020)

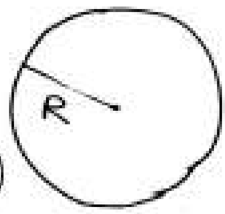
Problem based on Electrostatic Energy.

Q① Find the energy of a uniformly charged spherical shell of total charge q and radius R.

Sol.

Method 1

We have $W = \frac{1}{2}\int \sigma V \, ds$



Potential at surface

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{R} \quad \text{(constant)}$$

so

$$W = \frac{1}{2}\int\left(\sigma \times \frac{1}{4\pi\epsilon_0}\frac{q}{R}\right) ds$$

$$= \frac{1}{2}\times\frac{1}{4\pi\epsilon_0}\frac{q}{R}\underbrace{\int \sigma \, ds}$$

$$= \frac{1}{2}\times\frac{1}{4\pi\epsilon_0}\frac{q}{R}\times q$$

i.e $\boxed{W = \frac{q^2}{8\pi\epsilon_0 R}}$

Method 2:

We have

$$\vec{E} = \begin{cases} 0 & \text{inside} \\ \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} & \text{outside} \end{cases}$$

for outside $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$

$$E^2 = \vec{E}\cdot\vec{E} = \frac{1}{(4\pi\epsilon_0)^2}\frac{q^2}{r^4}$$

$$W_{tot} = \frac{\epsilon_0}{2}\int_{outside} E^2 \, d\tau = \frac{\epsilon_0}{2(4\pi\epsilon_0)^2}\int_{outside}\left(\frac{q^2}{r^4}\right) r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$W_{tot} = \frac{\epsilon_0}{2(4\pi\epsilon_0)^2} \times 4\pi \times q^2 \int_R^\infty \frac{1}{r^4} \cdot \cancel{r^2}\, dr$$

$$= \frac{\cancel{(4\pi\epsilon_0)} \times q^2}{(4\pi\epsilon_0)^{\cancel{2}} \times 2} \left[-\frac{1}{r}\right]_R^\infty$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q^2}{2} \left[-\frac{1}{\infty} + \frac{1}{R}\right]$$

$$\boxed{W_{tot} = \frac{1}{8\pi\epsilon_0} \frac{q^2}{R}}$$

Q. Find the energy stored in a uniformly charged solid sphere of radius R and charge q. Do it three different ways:

a) Use equation $W = \frac{1}{2}\int \rho V d\tau$

b) use equation $W = \frac{\epsilon_0}{2}\int E^2 d\tau$

c) use equation $W = \frac{\epsilon_0}{2}\left(\int_V E^2 d\tau + \oint V\vec{E}\cdot ds\right)$

Sol: We have calculated in the previous section.

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q r}{R^3} \hat{r} \quad \text{inside}$$

$$V = \frac{Q}{4\pi\epsilon_0} \frac{1}{2R}\left(3 - \frac{r^2}{R^2}\right)$$

and

$$\left.\begin{aligned} \vec{E} &= \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r} \\ V &= \frac{1}{4\pi\epsilon_0} \frac{Q}{r} \end{aligned}\right\} \text{outside.}$$

a) $W = \frac{1}{2}\int \rho V \, d\tau$

$$= \frac{1}{2}\rho \frac{1}{4\pi\epsilon_0}\frac{q}{2R}\int_0^R \left(3 - \frac{r^2}{R^2}\right) 4\pi r^2 \, dr$$

$$= \frac{q\rho \times 4\pi}{(4\pi\epsilon_0)\, 4R}\int_0^R \left(3r^2 - \frac{r^4}{R^2}\right) dr$$

$$= \frac{q\rho \times \cancel{4\pi}}{\cancel{4\pi}\epsilon_0 \times 4R}\left[\frac{\cancel{3}r^3}{\cancel{3}} - \frac{r^5}{5R^2}\right]_0^R$$

$$= \frac{q\rho}{\epsilon_0}\times\frac{1}{4R}\left[R^3 - \frac{R^{\cancel{5}\,3}}{5\cancel{R^2}}\right]$$

$$= \frac{q\rho}{\epsilon_0}\times\frac{1}{\cancel{4}R}\times R^3 \times \frac{\cancel{4}}{5}$$

$$= \frac{q\rho}{5\epsilon_0}\times R^2 = \left(\frac{qR^2}{5\epsilon_0}\right)\times\frac{q}{\frac{4}{3}\pi R^3} = \frac{1}{4\pi\epsilon_0}\left(\frac{3}{5}\frac{q^2}{R}\right)$$

i.e $\boxed{W = \frac{1}{4\pi\epsilon_0}\frac{3}{5}\left(\frac{q^2}{R}\right)}$

b) We have $\vec{E} = \begin{cases} \frac{1}{4\pi\epsilon_0}\left(\frac{qr}{R^3}\right)\hat{r} & \text{inside } (r<R) \\ \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} & \text{outside } (r>R) \end{cases}$

$$W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 \, d\tau$$

$$= \frac{\epsilon_0}{2}\left[\int_R^{\infty} E^2 \, d\tau + \int_0^R E^2 \, d\tau\right]$$

$$= \frac{\epsilon_0}{2}\left[\int_R^{\infty}\left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{q^2}{r^4}\, d\tau + \int_0^R \left(\frac{1}{4\pi\epsilon_0}\right)^2\left(\frac{qr}{R^3}\right)^2 d\tau\right.$$

$$W = \frac{\epsilon_0}{2} \times \left(\frac{q}{4\pi\epsilon_0}\right)^2 \cdot \left[\int_R^{\infty} \frac{1}{r^4} \times 4\pi r^2\, dr + \int_{\infty}^{R} \frac{r^2}{R^6} \times 4\pi r^2\, dr\right]$$

$$= \frac{\epsilon_0}{2} \times \left(\frac{q}{4\pi\epsilon_0}\right)^2 \times 4\pi \left[\int_R^{\infty} \frac{1}{r^2}\, dr + \frac{1}{R^6}\int_0^R r^4\, dr\right]$$

$$= \frac{\epsilon_0}{2} \times \left(\frac{q}{4\pi\epsilon_0}\right)^2 \times 4\pi \left[\left(-\frac{1}{r}\right)_R^{\infty} + \frac{1}{R^6}\left(\frac{r^5}{5}\right)_0^R\right]$$

$$= \frac{\cancel{\epsilon_0}}{2} \times \frac{q^2 \times 4\pi}{(4\pi)^2 \epsilon_0^2} \left[-\frac{1}{\infty} + \frac{1}{R} + \frac{1}{R^6}\left(\frac{R^5}{5} + \frac{0}{5}\right)\right]$$

$$= \frac{q^2}{8\pi\epsilon_0}\left[\frac{1}{R} + \frac{1}{5R}\right]$$

$$= \frac{q^2}{\cancel{8}_4\, \pi\epsilon_0} \times \frac{\cancel{6}^{\,3}}{5R} \Rightarrow \boxed{W = \frac{q^2}{4\pi\epsilon_0} \times \frac{3}{5R}}$$

6) We have to use equation

$$W = \frac{\epsilon_0}{2}\left(\int_V E^2\, d\tau + \oint_S V\vec{E} \cdot ds\right)$$

$$= \frac{\epsilon_0}{2}\left[\int_{r=0}^{R} E^2\, d\tau + \int_R^a \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)^2 (4\pi r^2\, dr)\right.$$

$$\left. + \oint_{r=a} \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r}\right)\left(\frac{1}{4\pi\epsilon_0}\frac{q}{\cancel{r^2}}\right)\cancel{r^2} \sin\theta\, d\theta\, d\phi\right]$$

$$= \frac{\epsilon_0}{2}\left[\int_0^R \frac{1}{(4\pi\epsilon_0)^2}\left(\frac{qr}{R^3}\right)^2 4\pi r^2\, dr + \int_R^a \left(\frac{1}{4\pi\epsilon_0}\right)^2 \cdot \frac{q^2}{r^4} \times 4\pi r^2\, dr\right.$$

$$\left. + \left(\left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{q^2}{r} \times 4\pi\right)_{r=a}\right]$$

(5)

$$W = \frac{\epsilon_0}{2}\left[\left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{q^2}{R^6} \times 4\pi \int_0^R r^4 dr + \left(\frac{1}{4\pi\epsilon_0}\right)^2 \times q^2 \times 4\pi \int_R^a \frac{1}{r^2} dr\right.$$

$$\left. + \left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{q^2}{a} \times 4\pi\right]$$

$$W = \frac{\epsilon_0}{2} \times \left(\frac{1}{4\pi\epsilon_0}\right)^2 \times q^2 \times 4\pi \left[\frac{1}{R^6} \times \left(\frac{R^5}{5}\right)_0^R + \left(-\frac{1}{r}\right)_R^a + \frac{1}{a}\right]$$

$$= \frac{1}{2} \times \frac{1}{4\pi\epsilon_0} \times q^2 \left[\frac{1}{R^5}\left(\frac{R^5}{5}\right) - \cancel{\frac{1}{a}} + \frac{1}{R} + \cancel{\frac{1}{a}}\right]$$

$$= \frac{1}{2} \times \frac{1}{4\pi\epsilon_0} \times q^2 \left[\frac{1}{5R} + \frac{1}{R}\right]$$

$$\boxed{W = \frac{1}{4\pi\epsilon_0} \times \frac{3}{5} \frac{q^2}{R}}$$

Note: A sphere of radius $a > R$ was used for calculation. and $V = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$

$$E = \frac{1}{4\pi\epsilon_0} \frac{qr}{R^3} \hat{r} \quad \text{inside}$$

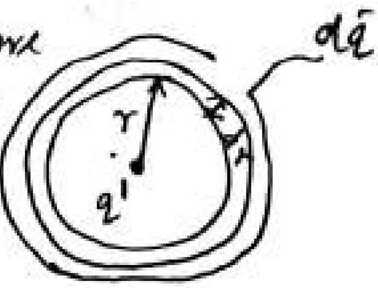
$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} \quad \text{outside}$$

Prob 2.33

The work done

$dW = dq' V$ → charge on sphere of radius r

$$dW = dq' \left(\frac{1}{4\pi\epsilon_0} \frac{q'}{r}\right)$$



The charge $q' = \frac{4}{3}\pi r^3 \rho$

$$= \frac{4}{3}\pi r^3 \times \frac{q}{\frac{4}{3}\pi R^3}$$

$$q' = \frac{q r^3}{R^3}$$

Similarly The charge $\because q' = \frac{4}{3}\pi r^3 \rho$

$$dq' = (4\pi r^2 dr)\rho$$

$$= (4\pi r^2 dr) \times \frac{q}{\frac{4}{3}\pi R^3}$$

$$dq' = \left(\frac{3r^2}{R^3}\right) q \, dr$$

Now $dW = \left(\frac{3r^2}{R^3}\right) q \cdot \left(\frac{1}{4\pi\epsilon_0}\right) \frac{1}{r} \times q \frac{r^3}{R^3} dr$

$$dW = \frac{1}{4\pi\epsilon_0} \frac{3q^2}{R^6} r^4 dr$$

$$W = \frac{1}{4\pi\epsilon_0} \frac{3q^2}{R^6} \int_0^R r^4 dr$$

$$= \frac{1}{4\pi\epsilon_0} \frac{3q^2}{R^6} \left[\frac{r^5}{5}\right]_0^R$$

$$\boxed{W = \frac{1}{4\pi\epsilon_0} \left(\frac{3q^2}{5R}\right)}$$

<u>Note</u>: In all above method, the energy or work done is same.

Comments on electrostatic energy

* A perplexing "Inconsistency"

Equation $W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 d\tau$ — ① clearly implies that the energy of stationary charge distribution is always positive. While the equation

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(r_i) \quad \text{②}$$

, can be positive or negative.

According to equation ②, the energy of two equal but opposite charge a distance r apart would be $-\frac{1}{4\pi\epsilon_0}\frac{q^2}{r}$. What" gone wrong? Which equation is correct?.

The answer is that Both equations are correct, but they pertain to slightly diff. situations. equation ① indicates that the energy of a point charge is infinite.

$$W = \frac{\epsilon_0}{2}\times\frac{1}{(4\pi\epsilon_0)^2}\int_0^\infty \frac{q^2}{r^4} r^2 \underbrace{\sin\theta\, d\theta\, d\phi}\, dr$$

$$= \frac{\epsilon_0}{2}\times\frac{1}{(4\pi\epsilon_0)^2}\times q^2 \times 4\pi\int_0^\infty \frac{1}{r^2}\, dr$$

$$= \frac{\epsilon_0}{2}\times\frac{1}{(4\pi\epsilon_0)^2}\times q^2\times 4\pi\left[-\frac{1}{0}+\frac{1}{\infty}\right]$$

$$\boxed{W = \infty}$$

Equation ① is more complete, in the sense that it tells you the total energy stored in charge configuration, but eq. ② is more appropriate when you are dealing with point charge.

* WHERE IS THE ENERGY STORED?

We have

$W = \frac{1}{2}\int \rho V \, d\tau$ and — over the charge distribution

$W = \frac{\epsilon_0}{2}\int E^2 \, d\tau$ — over the field.

For spherical shell

The charge confined to the surface and the electric field is present everywhere outside the surface. Where is the energy, then?

The energy being stored in the field, with a density in case of radiation theory

$\frac{\epsilon_0}{2}E^2 =$ energy per unit volume

But in electrostatics, one could just say "It is stored in the charge, with a density $\frac{1}{2}\rho V$".

* THE SUPERPOSITION PRINCIPLE

Because electrostatic energy is quadratic in the field, it does not obey the superposition principle.

The energy of a compound system is not the sum of the energies of its parts considered sepa rately - they are also "cross terms".

$$W_{tot} = \frac{\epsilon_0}{2}\int E^2 d\tau$$

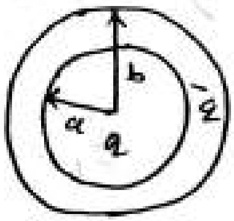
$$= \frac{\epsilon_0}{2}\int (\vec{E}_1+\vec{E}_2)^2 d\tau$$

$$= \frac{\epsilon_0}{2}\int (E_1^2+E_2^2+2\vec{E}_1\cdot\vec{E}_2) d\tau$$

$$W_{tot} = W_1 + W_2 + \epsilon_0\int \vec{E}_1\cdot\vec{E}_2\, d\tau$$

Ex: If you doble the charge everywhere, you quadropole the total energy.

Prob 2.34



Consider two concentric spherical shells, of radius a and b. Suppose the inner one carries a charge q, and the outer one a charge -q (both of them uniformly distributed over the surface). Calculate the energy ~~density~~ of this configurati.

a) using equation $W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 d\tau$

b) using equation $W_{tot} = W_1 + W_2 + \epsilon_0\int \vec{E}_1\cdot\vec{E}_2 d\tau$

and then result $W_{tot} = \frac{1}{8\pi\epsilon_0}\frac{q^2}{R}$.

$E = 0$ inside the sphere

$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$ outside the sphere.

Sol:

a) $W = \frac{\epsilon_0}{2}\int_{\text{allspace}} E^2 d\tau$. We have $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$ $(a<r<b)$

$= 0$ everywhere

$$= \frac{\epsilon_0}{2}\int_a^b \left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{q^2}{r^4}(4\pi r^2 dr)$$

$$= \frac{\epsilon_0}{2}\times\left(\frac{1}{4\pi\epsilon_0}\right)^2 \times q^2 \times 4\pi\int_a^b \frac{1}{r^2}dr$$

$$= \frac{\epsilon_0}{2}\times\left(\frac{1}{4\pi\epsilon_0}\right)^2 \times q^2 \times 4\pi\left[\frac{1}{a}-\frac{1}{b}\right]$$

$$\boxed{W = \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left(\frac{1}{a}-\frac{1}{b}\right)}$$

b) $W_1 = \frac{1}{4\pi\epsilon_0}\frac{q^2}{2a}$; $W_2 = \frac{1}{4\pi\epsilon_0}\frac{q^2}{2b}$

$$\vec{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \quad (r>a)$$

$$\vec{E}_2 = \frac{1}{4\pi\epsilon_0}\frac{-q\hat{r}}{r^2} \quad (r>b)$$

So $\vec{E}_1\cdot\vec{E}_2 = \left(\frac{1}{4\pi\epsilon_0}\right)^2\left(-\frac{q^2}{r^4}\right) \quad r>b$

So $\int_b^\infty E_1\cdot E_2\, d\tau = -\frac{1}{(4\pi\epsilon_0)^2}q^2\int_b^\infty \frac{1}{r^4}d\tau$

$$= -\frac{q^2}{(4\pi\epsilon_0)^2}\times 4\pi\int_b^\infty \frac{1}{r^{\cancel{4}2}}\cancel{r^2}\,dr = \frac{-q^2\times 4\pi}{(4\pi\epsilon_0)^2}\left[-\frac{1}{r}\right]_b^\infty$$

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The energy of a compound system is not the sum of the energies of its parts considered separately - they are also "cross terms".

$$W_{tot} = \frac{\epsilon_0}{2}\int E^2 \, d\tau$$

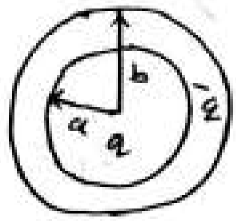
$$= \frac{\epsilon_0}{2}\int (\vec{E}_1 + \vec{E}_2)^2 \, d\tau$$

$$= \frac{\epsilon_0}{2}\int (E_1^2 + E_2^2 + 2\vec{E}_1 \cdot \vec{E}_2) \, d\tau$$

$$W_{tot} = W_1 + W_2 + \epsilon_0 \int \vec{E}_1 \cdot \vec{E}_2 \, d\tau$$

Ex. If you double the charge everywhere, you quadropole the total energy.

Prob 2.34



Consider two concentric spherical shells, of radius a and b. Suppose the inner one carries a charge q, and the outer one a charge $-q$ (both of them uniformly distributed over the surface). Calculate the energy ~~density~~ of this configuration.

a) using equation $W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 \, d\tau$

b) using equation $W_{tot} = W_1 + W_2 + \epsilon_0 \int \vec{E}_1 \cdot \vec{E}_2 \, d\tau$

and the result $W_{tot} = \frac{1}{8\pi\epsilon_0}\frac{q^2}{R}$.

$E = 0$ inside the sphere

$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$ outside the sphere.

Sol:

a) $W = \frac{\epsilon_0}{2}\int_{\text{allspace}} E^2 d\tau$, We have

$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$ ($a<r<b$)

$= 0$ everywhere

$$= \frac{\epsilon_0}{2}\int_a^b \left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{q^2}{r^4}(4\pi r^2 dr)$$

$$= \frac{\epsilon_0}{2}\times\left(\frac{1}{4\pi\epsilon_0}\right)^2 \times q^2 \times 4\pi \int_a^b \frac{1}{r^2} dr$$

$$= \frac{\epsilon_0}{2}\times\left(\frac{1}{4\pi\epsilon_0}\right)^2 \times q^2 \times 4\pi \left[\frac{1}{a}-\frac{1}{b}\right]$$

$$\boxed{W = \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left(\frac{1}{a}-\frac{1}{b}\right)}$$

b) $W_1 = \frac{1}{4\pi\epsilon_0}\frac{q^2}{2a}$; $W_2 = \frac{1}{4\pi\epsilon_0}\frac{q^2}{2b}$

$$\vec{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \quad (r>a)$$

$$\vec{E}_2 = \frac{1}{4\pi\epsilon_0}\frac{-q\hat{r}}{r^2} \quad (r>b)$$

So $\vec{E}_1\cdot\vec{E}_2 = \left(\frac{1}{4\pi\epsilon_0}\right)^2\left(-\frac{q^2}{r^4}\right) \quad r>b$

So $\int_b^\infty E_1\cdot E_2 \, d\tau = -\frac{1}{(4\pi\epsilon_0)^2} q^2 \int_b^\infty \frac{1}{r^4} d\tau$

$$= -\frac{q^2}{(4\pi\epsilon_0)^2}\times 4\pi\int_b^\infty \frac{1}{r^{\cancel{4}2}}\cancel{r^2}\, dr = -\frac{q^2\times 4\pi}{(4\pi\epsilon_0)^2}\left[-\frac{1}{r}\right]_b^\infty$$

$$\int \vec{E}_1 \cdot \vec{E}_2 \, d\tau = -\frac{q^2}{(4\pi\epsilon_0)^2} \times 4\pi \left[\frac{-1}{\infty} + \frac{1}{b} \right]$$

$$\int \vec{E}_1 \cdot \vec{E}_2 \, d\tau = -\frac{1}{(4\pi\epsilon_0)^2} \times \frac{q^2 \times 4\pi}{b}$$

Now

$$W_{tot} = W_1 + W_2 + \epsilon_0 \int \vec{E}_1 \cdot \vec{E}_2 \, d\tau$$

$$= \frac{1}{8\pi\epsilon_0} \frac{q^2}{a} + \frac{1}{8\pi\epsilon_0} \frac{q^2}{b} + \cancel{\epsilon_0} \times \frac{1}{(4\pi)^{\cancel{2}} \epsilon_0^{\cancel{2}}} \times -q^2 \times \cancel{4\pi} \times \frac{1}{b}$$

$$= \frac{q^2}{8\pi\epsilon_0} \left[\frac{1}{a} + \frac{1}{b} - \frac{2}{b} \right]$$

$$\boxed{W_{tot} = \frac{q^2}{8\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right)}$$