GE-4 | NUMERICAL DIFFERENTIATION

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* First - Order Derivaties
* Second - Order Derivatives

(i) Numerical Differentiation is the process of calculating the derivatives of a function by means of a set of given values of that function. The problem is solved by representing the function by an interpolation formula & then differentiating this formula as many times as desired.

(ii) If the function is given by a table of values for equidistant values of the independent variable, it should be represented by an interpolation formula employing differences, such as Newton's, Stirling's or Bessel's. But if the given values of the function are not for equidistant value of the independent variable, we must represent the function by Lagrange's or Hermite's formulas.

(iii) The considerations governing the choice of a formula employing differences are the same as in the case of interpolation. That is, if we desire the derivative at a point near the beginning of a set of tabular values, we use Newton's Forward formula. But if want derivaties at end of the table, we use Newton's Backward Formula. For points near the middle of the table, we should use a central-difference formula - Stirling's or Bessel's formula.

Let us consider Stirling's formula:- (2)

$$y = y_0 + u\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2}$$

$$+ \frac{u(u^2-1)(u^2-2^2)}{5!}\left(\frac{\Delta^5 y_{-3} + \Delta^5 y_{-2}}{2}\right)$$

$$+ \frac{u^2(u^2-1)(u^2-2^2)}{6!}\Delta^6 y_{-3} + \cdots$$

Now, $u = \frac{x-x_0}{h}$ & $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

$= \frac{1}{h}\cdot\frac{dy}{du}$

Hence,
$$\frac{dy}{dx} = \frac{1}{h}\left[\frac{\Delta y_{-1} + \Delta y_0}{2} + u\Delta^2 y_{-1} + \left(\frac{3u^2-1}{3!}\right)\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \left(\frac{4u^3-2u}{4!}\right)\Delta^4 y_{-2} + \left(\frac{5u^4-15u^2+4}{5!}\right)\left(\frac{\Delta^5 y_{-3} + \Delta^5 y_{-2}}{2}\right) + \frac{6u^5-20u^3+8u}{6!}\Delta^6 y_{-3} + \cdots\right] \quad —(1)$$

Similarly,

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_{-1} + u\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + u\left(\frac{\Delta^3 y_{2} + \Delta^3 y_{-1}}{2}\right) + \left(\frac{12u^2-2}{4!}\right)\Delta^4 y_{-2} + \frac{(20u^3-30u)}{5!}\left(\frac{\Delta^5 y_{-3} + \Delta^5 y_{-2}}{2}\right)\right. \quad —(2)$$

$$+ \left(\frac{30u^4 - 60u^2 + 8}{6!} \right) \Delta^6 y_{-3} + \cdots \Bigg] \quad (3)$$

Now, for the point $x = x_0$, we have $u = 0$. Hence, on substituting this value of u in the formulas above, we get

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) - \frac{1}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \frac{4}{5!}\left(\frac{\Delta^5 y_{-3} + \Delta^5 y_{-2}}{2}\right) + \cdots\right] \quad (3)$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{8}{6!}\Delta^6 y_{-3} + \cdots\right] \quad (4)$$

Similarly, one can find derivatives in exactly the same way by differentiating Newton's, Bessel's & Lagrange's formulas.

<u>Example</u> Find the first & second derivatives of the function tabulated below :- at pt $x = 0.6$.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.4	1.5836494				
		2137932			
0.5	1.7974426		330018		
				34710	
		2467950			3648
0.6	2.0442376		364728		
				38358	
		2832678			
0.7	2.3275054		403086		
		3235764			
0.8	2.6510818				

(4)

Soln: Here $x_0 = 0.6$, $u = 0$, $h = 0.1$. Also,

x	y	Δy $(\times 10^{-7})$	$\Delta^2 y$ $(\times 10^{-7})$	$\Delta^3 y$ $(\times 10^{-7})$	$\Delta^4 y$ $(\times 10^{-7})$
$0.4 = x_{-2}$	$1.5836494 = y_{-2}$				
		$2137932 = \Delta y_{-2}$			
$0.5 = x_{-1}$	$1.7974426 = y_{-1}$		$330018 = \Delta^2 y_{-2}$		
		$2467950 = \Delta y_{-1}$		$34710 = \Delta^3 y_{-2}$	
$0.6 = x_0$	$2.0442376 = y_0$		$364728 = \Delta^2 y_{-1}$		$3648 = \Delta^4 y_{-2}$
		$2832678 = \Delta y_0$		38358 $\Delta^3 y_{-1}$	
$0.7 = x_1$	$2.3275054 = y_1$		$403086 = \Delta^2 y_0$		
		$3235764 = \Delta y_1$			
$0.8 = x_2$	$2.6510818 = y_2$				

Hence substituting these values in Stirling's formula's derivative (as we are finding derivative in the middle of the table), we get,

[Using formulae (3) & (4)]

$$\left(\frac{dy}{dx}\right)_{x_0=0.6} = 10\left[\left(\frac{2467950 + 2832678}{2}\right)\times 10^{-7} - \frac{1}{6}\left(\frac{34710 + 38358}{2}\right)\times 10^{-7}\right]$$

$$= 10\left[0.2650314 - 0.0006089\right]$$

$$= 2.644225$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_0=0.6} = \frac{1}{(0.1)^2}\left[364728 - \frac{1}{12}\times 3648\right]\times 10^{-7}$$

$$= 100\left[0.0364728 - 0.0000304\right]\times 10^{-7}$$

$$= 3.64424$$