"Vector Valued Function"

Vector Valued function (Definition):- A vector valued function in 3-space is a function of the form

$$r(t) = f(t)i + g(t)j + h(t)k$$

where the function $f(t)$, $g(t)$, and $h(t)$ are real valued function of the parameter t. These functions are called component function or simply the components of the vector function $r(t)$.

vector valued functions are sometimes denoted as

$$\boxed{r(t) = \langle f(t), g(t), h(t) \rangle}.$$

Domain of vector valued function:- The domain of a vector valued function $r(t) = f(t)i + g(t)j + h(t)k$ is the set of all t's for which all the components $f(t)$, $g(t)$, $h(t)$ are defined.

Example 1 Determine the domain of the function $r(t) = ti + 2tj + e^t k$

Solution:- Given $r(t) = ti + 2tj + e^t k$

$\Rightarrow f(t) = t,\ g(t) = 2t,\ h(t) = e^t$

Now $f(t) = t$ is defined for all t's ie $\forall\ t \in \mathbb{R}$

$g(t) = 2t$ is defined for all t's ie $\forall\ t \in \mathbb{R}$

$h(t) = e^t$ is defined $\forall\ t \in \mathbb{R}$

$\Rightarrow r(t)$ is defined for all t's ie $\forall\ t \in \mathbb{R}$

$\Rightarrow$ Domain of $r(t) = ti + 2tj + e^t k$ is $\mathbb{R}$. (set of real number)

<u>Example(2)</u>:- Determine the domain of function

$$r(t) = \cos t\, i + \sin t\, j + \log t\, \hat{k}$$

<u>Solution</u>:- Given $r(t) = \cos t\, i + \sin t\, j + \log t \cdot k$

i.e $f(t) = \cos t$, $g(t) = \sin t$, $h(t) = \log t$

Now $f(t) = \cos t$, is defined, $\forall\, t \in \mathbb{R}$

$g(t) = \sin t$ is defined, $\forall\, t \in \mathbb{R}$

$h(t) = \log t$ is defined $\forall\, t \in (0, \infty)$

$\Rightarrow$ Domain of $r(t)$ is $\mathbb{R} \cap \mathbb{R} \cap (0, \infty) = (0, \infty)$.

<u>Example (3)</u>:- Determine the domain of vector valued function

$$r(t) = \cos t\, i + \log(4-t)\, j + \sqrt{t+2}\, k$$

<u>Solution</u>: Given $f(t) = \cos t$, $g(t) = \log(4-t)$, $h(t) = \sqrt{t+2}$

Now

(i) Domain $f(t) = \cos t$ is defined $\forall\, t \in \mathbb{R}$

(ii) $g(t) = \log(4-t)$ is defined if

if $(4-t) > 0 \Rightarrow$ ~~4>t~~ $4 > t$

$\Rightarrow g(t)$ is defined on $(4, \infty)$

(iii) $h(t) = \sqrt{t+2}$ is defined

if $(t+2) \geq 0 \Rightarrow t \geq -2$

$\Rightarrow h(t)$ is defined on $[-2, \infty)$

Hence Domain of $r(t)$ is $\mathbb{R} \cap (4, \infty) \cap [2, \infty) = (4, \infty)$

Graph of vector valued function:- The Graph of vector valued function $r(t) = f(t)i + g(t)j + h(t)K$ is defined to be the parametric curve described by the component of function of $r(t)$. OR

If $r(t) = f(t)i + g(t)j + h(t)K$, then the parametric curve represented by $x(t) = f(t)$, $y(t) = g(t)$, $z(t) = h(t)$ is called the graph of $r(t)$.

Example:- Draw the Graph of vector valued function

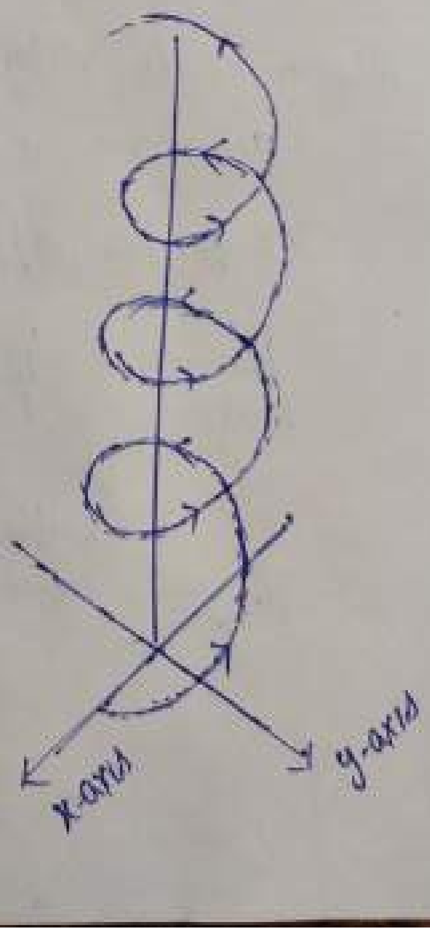
$r(t) = a\cos t\, i + a\sin t\, j + ctK$, where a, c is positive constant

Solution: Given $x(t) = a\cos t$, $y(t) = a\sin t$, $z(t) = c.t$

The first two component satisfy the cylinder's Equation

$$x^2+y^2 = (a\cos t)^2 + (a\sin t)^2$$

$$\boxed{x^2+y^2 = a^2}$$



which means that the Graph lies on the surface of the cylinder of radius a centred on the z-axis.

Now, as t increase, the z-component $z = ct$ also increase which means that the point (x, y, c) on the graph move upward.

Limit and continuity of vector valued function:-

Limit of vector valued function:- Let $r(t) = f(t)i + g(t)j + h(t)k$ be a vector valued function that is defined for all t in some open interval containing 'a' except that $r(t)$ need not be defined at 'a'. We say that $r(t)$ approaches the limit L (a vector) as t approaches a and write

$$\lim_{t \to a} r(t) = L$$

if, for every number $\varepsilon > 0$, there exist a corresponding

Theorem: number $\delta > 0$ s.t

$$\|r(t) - L\| < \varepsilon \text{ whenever } 0 < |t-a| < \delta.$$

Theorem:- A vector valued function $r(t) = \langle f(t), g(t), h(t) \rangle$ approaches a limit as $t \to a$ if and only if each components approaches a limit, and in this case

$$\lim_{t \to a} r(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle.$$

$$= \lim_{t \to a} f(t)i + \lim_{t \to a} g(t)j + \lim_{t \to a} h(t)k.$$

Example (1): If $r(t) = t\,i + (t^2+1)\,j + (2t+3)\,K$ Then find

$$\lim_{t\to 1} r(t).$$

Solution: We Know

$$\lim_{t\to a} r(t) = \lim_{r\to a} f(t)\,i + \lim_{r\to a} g(t)\,j + \lim_{r\to a} h(t)\,K$$

$$\Rightarrow \lim_{t\to 1} r(t) = \lim_{r\to 1} t\cdot i + \lim_{r\to 1} (t^2+1)\,j + \lim_{r\to 1} (2t+3)$$

$$= 1\cdot i + (1+1)\,j + (2\cdot 1+3)\,K$$

$$\Rightarrow \boxed{\lim_{t\to 1} r(t) = i + 2j + 5K}$$

Example (2): If $r(t) = \cos t\, i + \sin t\, i + (t^2+1)\,K$ then find the

$$\lim_{t\to \pi/4} r(t).$$

Solution: We Know, $\lim_{t\to a} r(t) = \lim_{t\to a} f(t)\,i + \lim_{t\to a} g(t)\,j + \lim_{t\to a} h(t)\,K$

$$\Rightarrow \lim_{t\to \pi/4} r(t) = \lim_{t\to \pi/4} \cos t\; i + \lim_{t\to \pi/4} \sin t\; j + \lim_{t\to \pi/4} (t^2+1)\,K$$

$$= \frac{1}{\sqrt{2}}\,i + \frac{1}{\sqrt{2}}\,j + \left[\left(\frac{\pi}{4}\right)^2+1\right]K$$

$$\Rightarrow \boxed{\lim_{t\to \pi/4} r(t) = \frac{1}{\sqrt{2}}\,i + \frac{1}{\sqrt{2}}\,j + \left(\frac{\pi^2}{16}+1\right)K}$$ Ans

Example (7):- Evaluate the limit $\lim_{t \to 0} \langle e^{2t}, \log(t+1), 2t+4 \rangle$

Solution: $\lim_{t \to 0} \langle e^{2t}, \log(t+1), (2t+4) \rangle$

$= \langle \lim_{t \to 0} e^{2t}, \lim_{t \to 0} \log(t+1), \lim_{t \to 0} (2t+4) \rangle$

$= \langle 1, \log(0+1), (2 \cdot 0+4) \rangle$

$= \langle 1, 0, 4 \rangle$

$\Rightarrow \boxed{\lim_{t \to 0} \langle e^{2t}, \log(t+1), (2t+4) \rangle = \langle 1, 0, 4 \rangle}$

Continuity of a vector valued function:- A vector valued function $r(t) = f(t)i + g(t)j + h(t)k$ is continuous at a point $t = a$ if $\boxed{\lim_{t \to a} r(t) = r(a)}$

i.e $\lim_{t \to a} r(t) = \lim_{r \to a} f(t)i + g(t)j + h(t)k$

$= f(a)i + g(a)j + h(a)k$

Example: i.e $\lim_{t \to a} [f(t)i + g(t)j + h(t)k]$

$= f(a)i + g(a)j + h(a)k$

Example (1):- Discuss the continuity of the vector valued function given by

$$r(t) = \cos t\, i + \sin t\, j + (t^2+1)K \quad \text{at } t=0$$

Solution: We have

$$\lim_{t\to a} r(t) = \lim_{t\to a} f(t)i + \lim_{t\to a} g(t)j + \lim_{t\to a} h(t)K$$

$$\Rightarrow \lim_{t\to 0} r(t) = \lim_{t\to 0} \cos t\, i + \lim_{t\to 0} \sin t\, j + \lim_{t\to 0} (t^2+1)K$$

$$= 1i + 0.j + 1K$$

$$\Rightarrow \boxed{\lim_{t\to 0} r(t) = i+K}$$

Example (2):- Discuss the continuity of the vector valued function

$$r(t) = e^t i + (1+\log(t+1))j + (1+\sin t) \quad \text{at } t=0$$

Solution: We have,

$$\lim_{t\to a} r(t) = \lim_{t\to a} f(t)i + \lim_{t\to a} g(t)j + \lim_{t\to a} h(t)K$$

$$\Rightarrow \lim_{t\to 0} r(t) = \lim_{t\to 0} e^t i + \lim_{t\to 0} [1+\log(t+1)]j + \lim_{t\to 0} (1+\sin t)K$$

$$= 1i + (1+\log 1)j + 1K$$

$$\Rightarrow \boxed{\lim_{t\to 0} r(t) = i+j+K}$$

Note: A vector valued function $r(t)$ is said to be continuous on an interval I if it is continuous at each point of I.

Example (1) Discuss the interval of continuity of vector valued function given by $r(t) = \cos 2t\, i + (1+\sin t)\, j + t^2 k$.

Solution:- $f(t) = \cos 2t$ is continuous on R

$g(t) = (1+\sin t)$ is continuous on R

$h(t) = t^2$ is continuous on R

Hence $r(t) = \cos 2t\, i + (1+\sin t)\, j + t^2 k$ is continuous on R.

Example: Discuss the interval of continuity of vector valued function given by

$$r(t) = e^t i + \log t\, j + \sqrt{t}\, k$$

Solution: $f(t) = e^t$ is continuous on $R = (-\infty, \infty)$

$g(t) = \log t$ is continuous on $(0, \infty)$

$h(t) = \sqrt{t}$ is continuous on $[0, \infty)$

Hence $r(t) = e^t i + \log t\, j + \sqrt{t}\, k$ is continuous on interval $(-\infty, \infty) \cap (0, \infty) \cap [0, \infty) = (0, \infty)$

Question: Find the interval of continuity of the following vector valued function.

(i) $r(t) = (1 + \sin t)\,i + \cos^2 t\, j + \log t\, K$

(ii) $r(t) = \left(\frac{\sin t}{t}\right) i + (\cos t + (t^2+1))\,j + e^t\, K$

(iii) $r(t) = (t^2+1)\,i + \sin t \cdot j + \frac{e^t}{(t-1)} \cdot K$

(iv) $r(t) = (t^4 + t^2 + 1)\,i + \frac{\sin t}{(t-1)(t+1)}\, j + \frac{\cos t}{\sin t} \cdot K.$

(v) $r(t) = \sqrt{t+1} \cdot i + \log(t+9)\, j + \frac{\tan t}{(t-1)} \cdot K.$