FOURIER'S AND LEPLACE'S INTEGRAL TRANSFORMS ①

In mathematical physics we frequently encounter pairs of functions related by an expression of the form

$$g(\alpha) = \int_a^b f(x)\, K(\alpha, x)\, dx$$

The function $g(\alpha)$ is called the <u>integral transform</u> of $f(x)$, by the kernel $K(\alpha, x)$. The integral transforms are very useful in mathematical analysis and physical applications.

There are a large number of different kinds of integral transforms depending on the choice of the kernel $K(\alpha, x)$ and the range of integration. The transforms of the function $f(x)$ for the kernels $e^{-i\alpha x}$, $e^{-\alpha x}$, $x J_n(\alpha x)$, $x^{\alpha-1}$ are called Fourier, Laplace, Hankel and Melin's transforms resp. i.e

$$g(\alpha) = \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx \qquad \text{(Fourier Transform)}$$

$$g(\alpha) = \int_0^{\infty} f(x) e^{-\alpha x} dx \qquad \text{(Laplace Transform)}$$

$$g(\alpha) = \int_0^{\infty} f(x)\, x\, J_n(\alpha x)\, dx \qquad \text{(Hankel Transform)}$$

$$g(\alpha) = \int_0^{\infty} f(x)\, x^{\alpha-1} dx \qquad \text{(Mellin Transform)}$$

We will however discuss only Fourier and Laplace integral transforms since they are specially useful in physical applications.

FOURIER TRANSFORM

If $f(x)$ is periodic function of x, then the Fourier integral of $f(x)$ may be expressed as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega x} d\omega \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t} dt \quad \text{— (A)}$$

This may be expressed as

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\omega x} g(\omega)\, d\omega \quad \text{— (1)}$$

where $g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$ — (2) or $g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$

The function $g(\omega)$ is called the <u>Fourier transform</u> of $f(t)$ and $f(t)$ is called <u>Fourier inverse transform</u> of $g(\omega)$. The integral (2) transforms a time function $f(t)$ into its equivalent frequency function $g(\omega)$, while integral ① reverses the process.

Infinite Fourier Sine and Cosine Transforms

The Fourier Transform of $f(t)$ is given by

$$g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} f(t)e^{-i\omega t}dt + \int_{0}^{\infty} f(t)e^{-i\omega t}dt\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{0}^{\infty} f(-t)\, e^{i\omega t}dt + \int_{0}^{\infty} f(t)\, e^{-i\omega t}dt\right] \quad — ③$$

(Replacing t by $-t$ in I^{st} integral)

Now $$f(t) = \begin{cases} f(-t) & \text{if function } f(t) \text{ is even} \\ -f(-t) & \text{if function } f(t) \text{ is odd} \end{cases} \quad — ④$$

Thus equation ③ gives

$$g(\omega) = \begin{cases} \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} f(t)\left(e^{i\omega t} + e^{-i\omega t}\right)dt & \text{for even function } f(t) \\ \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} f(t)\left(e^{-i\omega t} - e^{i\omega t}\right)dt & \text{for odd function } f(t) \end{cases}$$

Now, using $\cos\omega t = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$ and $\sin\omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$, we have

$$g(\omega) = \begin{cases} \frac{1}{\sqrt{2\pi}}\cdot 2\int_{0}^{\infty} f(t)\cos\omega t\, dt & \text{for even functions} \quad — ⑤ \\ \frac{1}{\sqrt{2\pi}}\cdot\frac{2}{i}\int_{0}^{\infty} f(t)\sin\omega t\, dt & \text{for odd functions} \quad — ⑥ \end{cases}$$

Infinite Fourier Sine and Cosine Transforms

Just as we have sine series representing odd functions and cosine series representing even functions, so we have sine and cosine Fourier integrals which represent odd or even functions resp. Let us prove that if $f(x)$ is odd, then $g(\omega)$ is odd too and show that in this case ① and ② reduces to a pair of sine transforms. The corresponding proof for even $f(x)$ is similar. Now

$$g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-i\omega x}dx$$

$$e^{-i\omega x} = \cos\omega x - i\sin\omega x$$

Then $$g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\{f(x)\cos\omega x - i f(x)\sin\omega x\}dx \quad — ③$$

Since $\cos\omega x$ is even and we are assuming that $f(x)$ is odd, the product $f(x)\cos\omega x$ is odd. The integral of an odd function over a symmetric interval about the origin (here, $-\infty$ to $+\infty$) is zero, so the term $\int_{-\infty}^{\infty} f(x)\cos\omega x$ in ③ is zero. The product $f(x)\sin\omega x$ is even (product of two odd functions) and therefore $\int_{-\infty}^{+\infty} f(x)\sin\omega x = 2\int_{0}^{\infty} f(x)\sin\omega x$. Substituting these into ③, we have

$$g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)(-i\sin\omega x)\,dx = -\frac{2i}{\sqrt{2\pi}}\int_{0}^{\infty} f(x)\sin\omega x\,dx$$

$$= -i\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\sin\omega x\,dx \qquad \text{— ④}$$

From ④, we can see that replacing ω by $-\omega$ changes the sign of $\sin\omega x$ and so changes the sign of $g(\omega)$. That is, $g(-\omega) = -g(\omega)$ or $g(\omega)$ is an odd function ~~above~~ Now expanding the exponential in ①, we find

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(\omega)e^{i\omega x}\,d\omega = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(\omega)(\cos\omega x + i\sin\omega x)\,d\omega$$

$$= \frac{2i}{\sqrt{2\pi}}\int_{0}^{\infty} g(\omega)\sin\omega x\,dx = i\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} g(\omega)\sin\omega x\,dx \qquad \text{— ⑤}$$

since $g(\omega)$ is odd

If we substituting $g(\omega)$ from ④ into ⑤ to obtain a Fourier integral (eqn ①), the numerical factor is $\left(-i\sqrt{\frac{2}{\pi}}\right)\left(i\sqrt{\frac{2}{\pi}}\right) = \frac{2}{\pi}$; thus the imaginary factors are not needed. The factor $\frac{2}{\pi}$ may multiply either of the two integrals or each integral may be multiplied by $\sqrt{\frac{2}{\pi}}$. We will make a latter choice in giving the following definitions

<u>Fourier Sine Transform</u> :- We define $f_s(x)$ and $g_s(\omega)$, a pair of <u>Fourier sine transforms</u> representing <u>odd functions</u>, by the equations

$$f_s(x) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} g_s(\omega)\sin\omega x\,d\omega$$

$$g_s(\omega) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f_s(x)\sin\omega x\,dx \qquad \text{— ⑥}$$

We can discuss even functions in a similar way and obtain

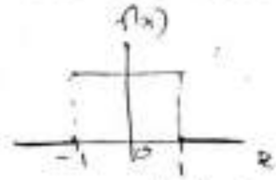
<u>Fourier Cosine Transform</u> :- We define $f_c(x)$ and $g_c(\omega)$, a pair of Fourier <u>cosine transforms</u> representing <u>even functions</u>, by the equations

$$f_c(x) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} g_c(\omega)\cos\omega x\,d\omega \qquad \text{— ⑦}$$

$$g_c(\omega) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f_c(x)\cos\omega x\,dx$$

Example:- Let us represent a non-periodic function as a Fourier integral. The function

$$f(x) = \begin{cases} 1, & -1 < x < 1 \\ 0, & |x| > 1 \end{cases}$$



might represent an impulse in mechanics (that is, a force applied only over a short time such as a bat hitting a baseball), or a sudden short surge of current in electricity, or a short pulse of sound or light which is not repeated. Since the given function is not periodic, it cannot be expanded in a Fourier series, since a Fourier series always represents a periodic function. Instead we write $f(x)$ as a Fourier integral. Using ②, we calculate $g(\omega)$; this process is like finding C_n's for a fourier series. We find

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left. \frac{e^{-i\omega x}}{-i\omega} \right|_{-1}^{1} = \frac{2}{\sqrt{2\pi}\,\omega} \, \frac{e^{-i\omega} - e^{i\omega}}{-2i} = \sqrt{\frac{2}{\pi}} \, \frac{\sin \omega}{\omega}$$

We subs $g(\omega)$ into the formula ① for $f(x)$ (this is like subs the evaluated coefficients into a Fourier series), we get

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(\omega) e^{i\omega x} d\omega = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\frac{2}{\pi}} \int_{-\infty}^{+\infty} \frac{\sin \omega}{\omega} e^{i\omega x} d\omega$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \omega}{\omega} e^{i\omega x} dx = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin \omega (\cos \omega x + i \sin \omega x)}{\omega} d\omega$$

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin \omega \cos \omega x}{\omega} d\omega \quad \text{— ⓐ}$$

Since $\frac{\sin \omega}{\omega}$ is an even function. We thus have an integral representing the function $f(x)$ shown in fig above.

We can use ⓐ to evaluate a definite integral.

We have

$$\int_{0}^{\infty} \frac{\sin \omega \cos \omega x}{\omega} d\omega = \frac{\pi}{2} f(x) = \begin{cases} \frac{\pi}{2} & \text{for } |x| < 1 \\ \frac{\pi}{4} & \text{for } |x| = 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

Notice that we have used the fact that the Fourier integral represents the midpoint of the jump in $f(x)$ at $|x| = 1$. If we let $x = 0$, we get

$$\int_0^\infty \frac{\sin \omega}{\omega} d\omega = \frac{\pi}{2}$$

Notice that we could have done this problem by observing that $f(x)$ is an even function and so can be represented by a cosine transform.

Properties of Fourier's Transform :-

① Addition theorem or Linearity Theorem :- If $f(t) = a_1 f_1(t)$

If $f(t) = a_1 f_1(t) + a_2 f_2(t) + \cdots$, then the Fourier transform of $f(t)$ is given by

$$g(\omega) = a_1 g_1(\omega) + a_2 g_2(\omega) + \cdots \quad —①$$

where $g_1(\omega), g_2(\omega)$ -- are Fourier transforms of $f_1(t), f_2(t)$ -- and a_1, a_2, -- are constants.

② Similarity Theorem or Change of Scale Property :- If $g(\omega)$ is the Fourier transform of $f(t)$, the Fourier transform of $f(at)$ is $\frac{1}{a} g\left(\frac{\omega}{a}\right)$

Proof :- Denoting the Fourier transform of $f(t)$ by $F.T.[f(t)]$, we have

$$F.T.[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt = g(\omega)$$

Hence $F.T.[f(at)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(at) e^{-i\omega t} dt$

Subs $y = at$ in above integral, we get

$$F.T.[f(at)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) e^{-i\omega \frac{y}{a}} \frac{dy}{a}$$

$$= \frac{1}{a} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) e^{-i\left(\frac{\omega}{a}\right) y} dy = \frac{1}{a} g\left(\frac{\omega}{a}\right) \quad —②$$

This theorem is well known in its applications to wave forms and spectra, where compression of time scale by given factor compresses the periods of all harmonic components equally and therefore increases the frequency of every component by the same factor.

③ If $g(\omega)$ is the Fourier transform of $f(t)$, then the Fourier transform of the complex conjugate of $f(t)$ will be given by $g^*(-\omega)$; where * indicates the complex conjugate of the corresponding complex function.

Proof:- We have

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t}\, dt$$

Taking complex conjugate of both sides, we get

$$g^*(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f^*(t)\, e^{i\omega t}\, dt$$

Replacing ω by $(-\omega)$, we get

$$g^*(-\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f^*(t)\, e^{-i\omega t}\, dt$$

Hence $g^*(-\omega) = F.T.[f^*(t)]$ — (3)

4. <u>Shifting Property</u>:- If $g(\omega)$ is the Fourier transform of $f(t)$; the Fourier transform of $f(t \pm a)$ will be given by $e^{\pm i\omega a} g(\omega)$; where a is any constant

Proof:- By definition of infinite Fourier transform

$$F.T.[f(t \pm a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t \pm a)\, e^{-i\omega t}\, dt$$

Subs $(t \pm a) = y$, i.e. $dt = dy$; we have

$$F.T.[f(t \pm a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y)\, e^{-i\omega (y \mp a)}\, dy$$

$$= e^{\pm i\omega a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y)\, e^{-i\omega y}\, dy$$

$$= e^{\pm i\omega a} g(\omega) \quad \text{— (4)}$$

According to this theorem if a given function be shifted in the positive or negative direction by an amount a, no Fourier component changes in amplitude; but its Fourier transform suffers phase change.

5. <u>Modulation Theorem</u>:- If $g(\omega)$ is the Fourier transform of $f(t)$, then Fourier transform of $f(t)\cos at$ is given by

$$\frac{1}{2} g(\omega - a) + \frac{1}{2} g(\omega + a)$$

Proof:- $F.T.[f(t)\cos at] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) \cos at\, e^{-i\omega t}\, dt$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) \left(\frac{e^{iat} + e^{-iat}}{2} \right) e^{-i\omega t}\, dt$$

$f_1(t)$ and $f_2(t)$ is given by an integral known as <u>convolution integral</u> where the functions g_1 and g_2 are said to convolve with each other. Also $g_1 * g_2 = g_2 * g_1$

7) Fourier transform of a Convolution (Convolution or Faulting theorem for Fourier transforms) ~~Parseval's theorem~~ :- The Fourier transform of a convolution integral is given by the product of transforms of the convolving functions.

Proof :- let $f_1 * f_2$ be given convolution integral, i.e.

$$f_1 * f_2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_1(t') f_2(t-t')\, dt' \quad \text{—(11)}$$

The Fourier transform of $f(t)$ is

$$F.T.[f_1 * f_2] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_1(t') f_2(t-t') e^{-i\omega t} dt'\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} f_1(t') e^{-i\omega t'} dt' \int_{-\infty}^{+\infty} f_2(t-t') e^{-i\omega t} e^{i\omega t'} dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} f_1(t') e^{-i\omega t'} dt' \int_{-\infty}^{+\infty} f_2(t-t') e^{-i\omega(t-t')} dt \quad \text{—(12)}$$

If $g_1(\omega)$ and $g_2(\omega)$ are Fourier transforms of $f_1(t)$ and $f_2(t)$ respectively, we have

$$g_1(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_1(t') e^{-i\omega t'} dt' \quad \text{—(13)}$$

$$g_2(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_2(t) e^{-i\omega t} dt \quad \text{—(14)}$$

changing t by $(t-t')$ in (14), we get

$$g_2(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_2(t-t') e^{-i\omega(t-t')} dt \quad \text{—(15)}$$

Hence from (12), (13) and (15), we have

$$F.T.[f_1 * f_2] = g_1(\omega) \cdot g_2(\omega) \quad \text{—(16)}$$

In other words,

$g_1 g_2$ and $f_1 * f_2$ are a pair of Fourier transforms. (1)

Because of the symmetry of the $f(x)$ and $g(\omega)$ integrals, there is a similar result relating

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-i(\omega-a)t} f(t) + e^{-i(\omega+a)t} f(t)dt\right]$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-i(\omega-a)t} f(t) + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-i(\omega+a)t} f(t)\,dt\right]$$

$$= \frac{1}{2}\left[g(\omega-a) + g(\omega+a)\right] \quad — (5)$$

(6) Convolution ~~Theorem~~ Integral :- The transform of a product of two functions is given by a convolution integral

Proof :- Let $f_1(t)$ and $f_2(t)$ be two given functions and their product functions $f(t)$ is $f(t) = f_1(t) f_2(t)$.

From definition $F.T.[f(t)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f_1(t) f_2(t) e^{-i\omega t} dt$ — (6)

If $g_1(\omega)$ is the Fourier transform of $f_1(t)$, then the Fourier inverse transform $g_1(\omega')$ is

$$f_1(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\omega' t} g_1(\omega') d\omega' \quad — (7)$$

Substituting value of $f_1(t)$ from (7) in (6), we get

$$F.T.[f_1(t)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g_1(\omega') e^{i\omega' t} d\omega'\right\} f_2(t) e^{-i\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left\{ g_1(\omega') \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f_2(t) e^{-(\omega-\omega')t} dt\right\} d\omega' \quad — (8)$$

Now the Fourier transform of $f_2(t)$ is given by

$$g_2(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f_2(t) e^{-i\omega t} dt$$

Replacing ω by $\omega - \omega'$ in above equation, we get

$$g_2(\omega-\omega') = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f_2(t) e^{-i(\omega-\omega')t} dt \quad — (9)$$

Combining (8) and (9), the Fourier transform of $f(t)$ becomes

$$F.T.[f(t)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g_1(\omega') g_2(\omega-\omega') d\omega' \quad — (10)$$

$$= g_1 * g_2$$

Thus the Fourier transform of a product of two functions and $g_1 = g_2 = g$, we have

$$\int_{-\infty}^{+\infty} |g(\omega)|^2 d\omega = \int_{-\infty}^{+\infty} |f(x)|^2 dx \quad -(21)$$

From (19), we have

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_1^*(x)\, f_2(x)\, e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g_1^*\left[-(\omega-\omega')\right] g_2(\omega')\, d\omega'$$

$\Rightarrow$ Fourier transform of the ~~sqy~~ squared modulus of a function is given by self convolution integral.

Derivatives of Fourier Transform :- If $g(\omega)$ is the Fourier transform of $f(t)$, then

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t} dt \quad - 22$$

Differentiating both sides of above equation w.r.t ω, we have

$$\frac{dg(\omega)}{d\omega} = \frac{1}{\sqrt{2\pi}} \frac{d}{d\omega} \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\partial}{\partial \omega}\left[f(t)\, e^{-i\omega t} \right] dt$$

$$= -i \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} t\, f(t)\, e^{-i\omega t} dt$$

$$\frac{dg(\omega)}{d\omega} = -i\, F.T.\left[t\, f(t) \right] \quad — (23)$$

If we differentiate (23) n times w.r.t ω, we get

$$\frac{d^n g(\omega)}{d\omega^n} = (-i)^n\, F.T.\left[t^n f(t) \right] \quad — (24)$$

Fourier transform of a Derivative :

Let $g_I(\omega)$ be the Fourier transform of the I^{st} derivative of a function $f(t)$, then

$$g_I(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{df}{dt}\, e^{-i\omega t} dt \quad — (1)$$

Integrating by parts, we get

$$g_I(\omega) = \frac{1}{\sqrt{2\pi}} \left[e^{-i\omega t} f(t) \right]_{-\infty}^{+\infty} + \frac{i\omega}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}}\left[e^{-i\omega t} f(t)\right]_{-\infty}^{+\infty} + i\omega g(\omega) \quad -(2)$$

where $g(\omega)$ is the Fourier transform of $f(t)$. Now as $t \to \infty$, $e^{-i\omega t} \to 0$ and as $t \to -\infty$, then $e^{-i\omega t} \to \infty$, therefore for the limit to exist, the function $f(t)$ should be well behaved function such that $f(t)$ must decrease to zero as $t \to -\infty$ at a much faster rate than at which $e^{-i\omega t}$ tends to infinity. Then

$$g_1(\omega) = i\omega g(\omega) \quad -(3)$$

i.e. $$F.T.\left[\frac{df}{dt}\right] = i\omega\, F.T.[f(t)] \quad -(4a)$$

or $$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \frac{df}{dt} e^{-i\omega t} dt = i\omega \cdot \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt \quad -(4b)$$

Replacing $f(t)$ by $\frac{df}{dt}$ on both sides of above equation, we get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \frac{d^2f}{dt^2} e^{-i\omega t} dt = i\omega \cdot \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \frac{df}{dt} e^{-i\omega t} dt$$

$$= (i\omega)^2 g(\omega)$$

i.e. $$F.T.\left[\frac{d^2f}{dt^2}\right] = (i\omega)^2 F.T.[f(t)] \quad -(5)$$

Repeating the process n times, we get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \frac{d^nf}{dt^n} e^{-i\omega t} dt = (i\omega)^n g(\omega)$$

or $$F.T.\left[\frac{d^nf}{dt^n}\right] = (i\omega)^n F.T.[f(t)] \quad -(6)$$

Fourier sine and cosine transforms of Derivatives:

The Fourier sine and cosine transforms of a function $f(t)$ are defined as

$$g_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\sin\omega t\, dt \quad -(1)$$

$$g_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\cos\omega t\, dt \quad -(2)$$

Now, let us assume a well behaved function $f(t)$ such that $f(t)$ and its derivatives $\to$ zero as $t \to \infty$