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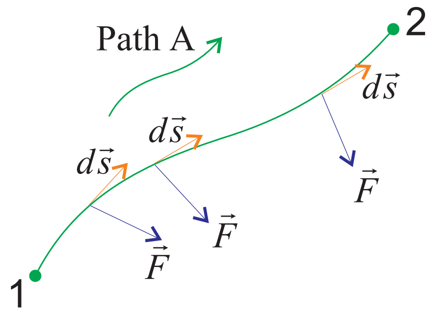
Subject: Electricity & Magnetism
Semester: II

Electric Potential

4.1 Potential Energy and Conservative Forces

(Read Halliday Vol.1 Chap.12)
Electric force is a **conservative force**



Work done by the electric force $\vec{F}$ as a charge moves an infinitesimal distance $d\vec{s}$ along $Path\ A = dW$

Note: $d\vec{s}$ is in the *tangent* direction of the curve of $Path\ A$.

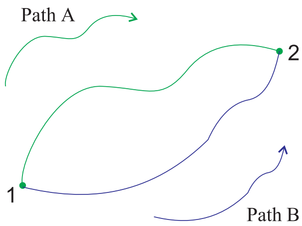
$$dW = \vec{F} \cdot d\vec{s}$$

$\therefore$ Total work done W by force $\vec{F}$ in moving the particle from Point 1 to Point 2

$$\boxed{W = \underset{\text{Path A}}{\int_1^2} \vec{F} \cdot d\vec{s}}$$

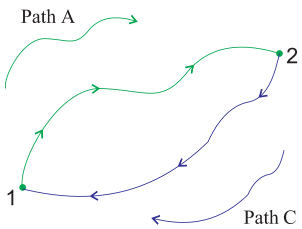
$$\begin{aligned} \underset{\text{Path A}}{\int_1^2} &= \textit{Path Integral} \\ &= \text{Integration over Path A from Point 1 to Point 2.} \end{aligned}$$

DEFINITION: A force is **conservative** *if* the work done on a particle by the force is *independent of the path taken.*



$\therefore$ For conservative forces,

$$\underset{\text{Path A}}{\int_1^2 \vec{F}\cdot d\vec{s}} = \underset{\text{Path B}}{\int_1^2 \vec{F}\cdot d\vec{s}}$$



Let's consider a path starting at point 1 to 2 through *Path A* and from 2 to 1 through *Path C*

$$\begin{aligned}\text{Work done} &= \underset{\text{Path A}}{\int_1^2 \vec{F}\cdot d\vec{s}} + \underset{\text{Path C}}{\int_2^1 \vec{F}\cdot d\vec{s}} \\ &= \underset{\text{Path A}}{\int_1^2 \vec{F}\cdot d\vec{s}} - \underset{\textbf{Path B}}{\int_{\mathbf{1}}^{\mathbf{2}} \vec{F}\cdot d\vec{s}}\end{aligned}$$

DEFINITION: The work done by a **conservative force** on a particle when it *moves around a closed path returning to its initial position* is *zero.*

MATHEMATICALLY, $\vec{\nabla}\times\vec{F} = 0$ everywhere for conservative force $\vec{F}$

Conclusion: Since the work done by a conservative force $\vec{F}$ is *path-independent*, we can define a quantity, **potential energy**, that depends only on the *position* of the particle.

Convention: We define **potential energy** U such that

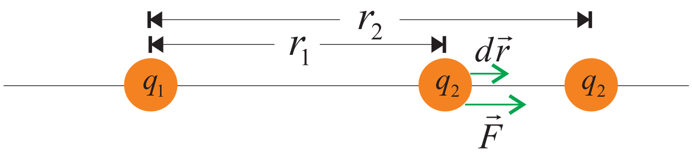
$$\boxed{dU = -W = -\int \vec{F}\cdot d\vec{s}}$$

$\therefore$ For particle moving from 1 to 2

$$\int_1^2 dU = U_2 - U_1 = -\int_1^2 \vec{F}\cdot d\vec{s}$$

where U_1, U_2 are **potential energy** at position 1, 2.

Example:



Suppose charge q_2 moves from point 1 to 2.

$$\begin{aligned}
\text{From definition:}\quad U_2 - U_1 &= -\int_1^2 \vec{F}\cdot d\vec{r} \\
&= -\int_{r_1}^{r_2} F\,dr \quad (\because\ \vec{F} \parallel d\vec{r}\,) \\
&= -\int_{r_1}^{r_2} \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\,dr \\
(\because \int \frac{dr}{r^2} = -\frac{1}{r} + C\,) \qquad &= \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r}\bigg|_{r_1}^{r_2} \\
-\Delta W = \Delta U &= \frac{1}{4\pi\epsilon_0} q_1 q_2 \left(\frac{1}{r_2} - \frac{1}{r_1}\right)
\end{aligned}$$

Note:

(1) This result is generally true for 2-Dimension or 3-D motion.

(2) If q_2 moves away from q_1,
then $r_2 > r_1$, we have

- If q_1, q_2 are of *same* sign,
  then $\Delta U < 0$, $\Delta W > 0$
  (ΔW = Work done by electric *repulsive* force)
- If q_1, q_2 are of *different* sign,
  then $\Delta U > 0$, $\Delta W < 0$
  (ΔW = Work done by electric *attractive* force)

(3) If q_2 moves towards q_1,
then $r_2 < r_1$, we have

- If q_1, q_2 are of *same* sign,
  then ΔU 0, ΔW 0
- If q_1, q_2 are of *different* sign,
  then ΔU 0, ΔW 0

(4) Note: It is the *difference* in potential energy that is important.

$$\text{REFERENCE POINT:} \quad U(r=\infty)=0$$

$$\therefore\ U_\infty - U_1 = \frac{1}{4\pi\epsilon_0}\, q_1 q_2 \left(\frac{1}{r_2} - \frac{1}{r_1}\right), \quad r_2 \to \infty$$

$$\boxed{U(r) = \frac{1}{4\pi\epsilon_0}\cdot\frac{q_1 q_2}{r}}$$

If q_1, q_2 *same sign*, then $U(r) > 0$ for all r

If q_1, q_2 *opposite sign*, then $U(r) < 0$ for all r

(5) Conservation of Mechanical Energy:
For a system of charges with no external force,

$$E = K + U = \text{Constant}$$

K (Kinetic Energy), U (Potential Energy)

or $\boxed{\Delta E = \Delta K + \Delta U = 0}$

<u>Potential Energy of A System of Charges</u>

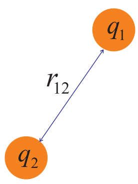
Example: P.E. of 3 charges q_1, q_2, q_3

Start: q_1, q_2, q_3 all at $r = \infty, U = 0$

Step1:

Move q_1 from ∞ to its position $\Rightarrow\ U = 0$

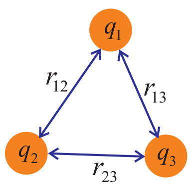
Step2:



Move q_2 from ∞ to new position $\Rightarrow$

$$U = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}}$$

Step3:



Move q_3 from ∞ to new position $\Rightarrow$ Total P.E.

$$U = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$$

Step4: What if there are 4 charges?

4.2 Electric Potential

Consider a charge q at center, we consider its effect on test charge q_0

DEFINITION: We define electric potential V so that

$$\boxed{\Delta V = \frac{\Delta U}{q_0} = \frac{-\Delta W}{q_0}}$$

$(\therefore\ V$ is the P.E. per unit charge$)$

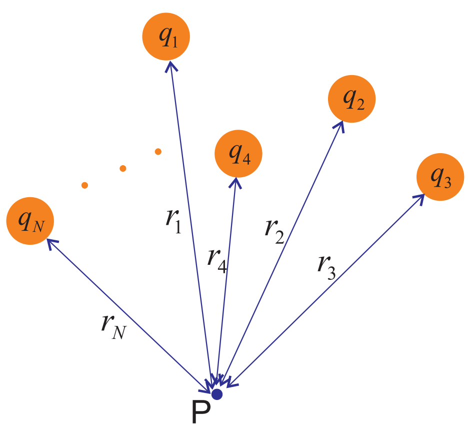
- Similarly, we take $V(r = \infty) = 0$.
- Electric Potential is a **scalar**.
- Unit: $Volt(V) = Joules/Coulomb$
- For a single point charge:

$$\boxed{V(r) = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r}}$$

- Energy Unit: $\Delta U = q\Delta V$

$$electron - Volt(eV) \;=\; \underbrace{1.6 \times 10^{-19}}_{\text{charge of electron}}\ J$$

Potential For A System of Charges



For a total of N point charges, the potential V at any point P can be derived from the **principle of superposition**.

Recall that potential due to q_1 at point P: $V_1 = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1}{r_1}$

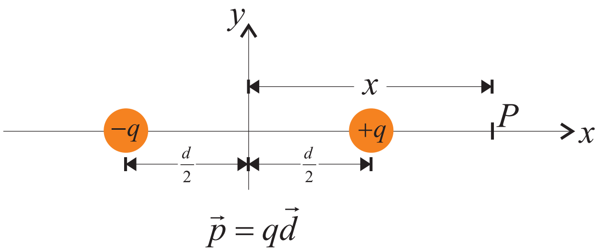
$\therefore$ Total potential at point P due to N charges:

$$\begin{aligned} V &= V_1 + V_2 + \cdots + V_N \quad \text{(principle of superposition)} \\ &= \frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{r_1} + \frac{q_2}{r_2} + \cdots + \frac{q_N}{r_N}\right] \end{aligned}$$

$$V = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i}$$

Note: For $\vec{E}, \vec{F}$, we have a sum of vectors
For V, U, we have a sum of scalars

Example: Potential of an electric dipole



Consider the potential of point P at distance $x > \frac{d}{2}$ from dipole.

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{+q}{x-\frac{d}{2}} + \frac{-q}{x+\frac{d}{2}}\right]$$

Special Limiting Case: $x \gg d$

$$\frac{1}{x \mp \frac{d}{2}} = \frac{1}{x}\cdot\frac{1}{1\mp\frac{d}{2x}} \simeq \frac{1}{x}\left[1 \pm \frac{d}{2x}\right]$$

$$\therefore \quad V = \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{x}\left[1+\frac{d}{2x} - (1-\frac{d}{2x})\right]$$

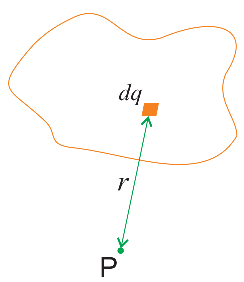
$$V = \frac{p}{4\pi\epsilon_0 x^2} \qquad \text{(Recall } p = qd\text{)}$$

For a point charge $E \propto \frac{1}{r^2}$ $V \propto \frac{1}{r}$

For a dipole $E \propto \frac{1}{r^3}$ $V \propto \frac{1}{r^2}$

For a quadrupole $E \propto \frac{1}{r^4}$ $V \propto \frac{1}{r^3}$

Electric Potential of Continuous Charge Distribution



For any charge distribution, we write the electrical potential dV due to infinitesimal charge dq:

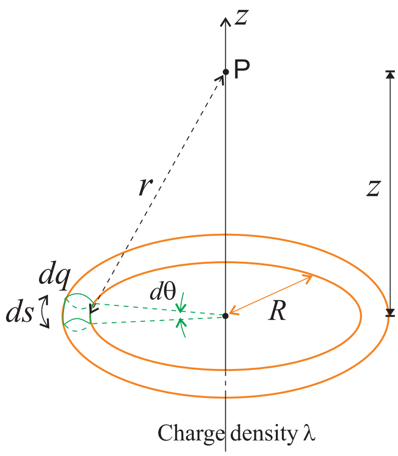
$$dV = \frac{1}{4\pi\epsilon_0}\cdot\frac{dq}{r}$$

$$\therefore \quad \boxed{V = \int_{\substack{\text{charge} \\ \text{distribution}}} \frac{1}{4\pi\epsilon_0} \cdot \frac{dq}{r}}$$

Similar to the previous examples on E-field, for the case of *uniform* charge distribution:

$$\begin{array}{lllll} \text{1-D} & \Rightarrow & \text{long rod} & \Rightarrow & dq = \lambda\, dx \\ \text{2-D} & \Rightarrow & \text{charge sheet} & \Rightarrow & dq = \sigma\, dA \\ \text{3-D} & \Rightarrow & \text{uniformly charged body} & \Rightarrow & dq = \rho\, dV \end{array}$$

Example (1): Uniformly-charged ring



Length of the infinitesimal ring element $= ds = Rd\theta$

$$\therefore \quad \text{charge } dq = \lambda\, ds = \lambda R\, d\theta$$

$$dV = \frac{1}{4\pi\epsilon_0} \cdot \frac{dq}{r} = \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda R\, d\theta}{\sqrt{R^2 + z^2}}$$

The integration is around the entire ring.

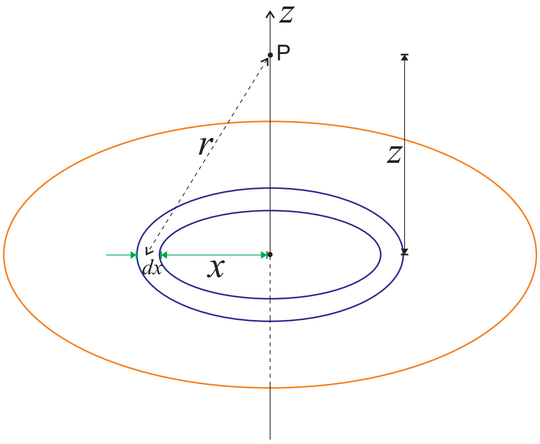
$$\begin{aligned} \therefore \quad V &= \int_{\text{ring}} dV \\ &= \int_0^{2\pi} \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda R\, d\theta}{\sqrt{R^2 + z^2}} \\ &= \frac{\lambda R}{4\pi\epsilon_0 \sqrt{R^2 + z^2}} \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \end{aligned}$$

Total charge on the ring $= \lambda \cdot (2\pi R)$

$$V = \frac{Q}{4\pi\epsilon_0 \sqrt{R^2 + z^2}}$$

LIMITING CASE: $z \gg R \;\Rightarrow\; V = \dfrac{Q}{4\pi\epsilon_0 \sqrt{z^2}} = \dfrac{Q}{4\pi\epsilon_0 |z|}$

Example (2): Uniformly-charged disk



Total charge = Q

Charge density = σ

Using the **principle of superposition**, we will find the potential of a disk of uniform charge density by integrating the potential of *concentric rings.*

$$\therefore \quad dV = \frac{1}{4\pi\epsilon_0}\int_{\text{disk}} \frac{dq}{r}$$

Ring of radius x: $dq = \sigma\, dA = \sigma\,(2\pi x dx)$

$$\begin{aligned}
\therefore \quad V &= \int_0^R \frac{1}{4\pi\epsilon_0}\cdot\frac{\sigma 2\pi x\, dx}{\sqrt{x^2+z^2}} \\
&= \frac{\sigma}{4\epsilon_0}\int_0^R \frac{d(x^2+z^2)}{(x^2+z^2)^{1/2}} \\
V &= \frac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+R^2}-\sqrt{z^2}\right) \\
&= \frac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+R^2}-|z|\right)
\end{aligned}$$

Recall:
$$|x| = \begin{cases} +x; & x \geq 0 \\ -x; & x < 0 \end{cases}$$

Limiting Case:

(1) If $|z| \gg R$

$$\begin{aligned}
\sqrt{z^2+R^2} &= \sqrt{z^2\left(1+\frac{R^2}{z^2}\right)} \\
&= |z|\cdot\left(1+\frac{R^2}{z^2}\right)^{\frac{1}{2}} \qquad (\ (1+x)^n \approx 1+nx \text{ if } x \ll 1\ ) \\
&\simeq |z|\cdot\left(1+\frac{R^2}{2z^2}\right) \qquad (\ \frac{|z|}{z^2} = \frac{1}{|z|}\ )
\end{aligned}$$

$\therefore$ At large z, $V \simeq \frac{\sigma}{2\epsilon_0}\cdot\frac{R^2}{2|z|} = \frac{Q}{4\pi\epsilon_0|z|}$ (like a point charge)
where Q = total charge on disk $= \sigma\cdot\pi R^2$

(2) If $|z| \ll R$

$$\begin{aligned}\sqrt{z^2+R^2} &= R\cdot\Big(1+\frac{z^2}{R^2}\Big)^{\frac{1}{2}} \\ &\simeq R\Big(1+\frac{z^2}{2R^2}\Big)\end{aligned}$$

$$\therefore \quad V \simeq \frac{\sigma}{2\epsilon_0}\Big[R-|z|+\frac{z^2}{2R}\Big]$$

At $z=0$, $V=\dfrac{\sigma R}{2\epsilon_0}$; Let's call this V_0

$$\begin{aligned}\therefore \quad V(z) &= \frac{\sigma R}{2\epsilon_0}\Big[1-\frac{|z|}{R}+\frac{z^2}{2R^2}\Big] \\ V(z) &= V_0\Big[1-\frac{|z|}{R}+\frac{z^2}{2R^2}\Big]\end{aligned}$$

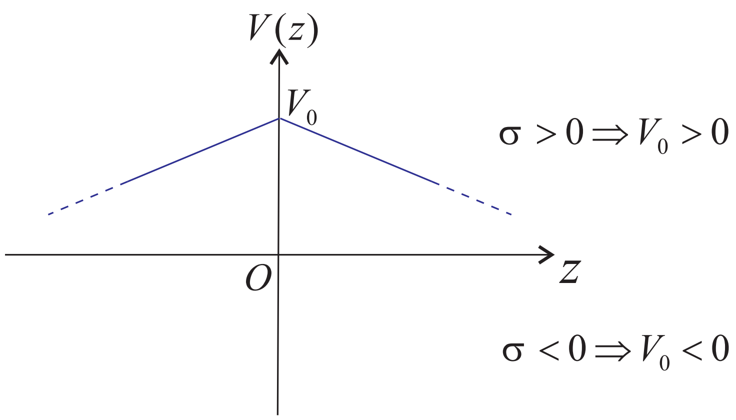
The *key* here is that it is the difference between potentials of two points that is important.

$\Rightarrow$ A convenience reference point to compare in this example is the potential of the charged disk.

$\therefore$ The important quantity here is

$$V(z)-V_0 = -\frac{|z|}{R}V_0 + \underbrace{\frac{z^2}{2R^2}V_0}_{\text{neglected as } z \ll R}$$

$$\boxed{V(z)-V_0 = -\frac{V_0}{R}|z|}$$



4.3 Relation Between Electric Field E and Electric Potential V

(A) To get V from $\vec{E}$:

Recall our definition of the potential V:

$$\Delta V = \frac{\Delta U}{q_0} = -\frac{W_{12}}{q_0}$$

where ΔU is the change in P.E.; W_{12} is the work done in bringing charge q_0 from point 1 to 2.

$$\therefore \quad \Delta V = V_2 - V_1 = \frac{-\int_1^2 \vec{F} \cdot d\vec{s}}{q_0}$$

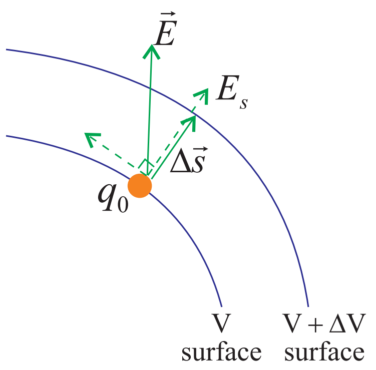
However, the definition of E-field: $\vec{F} = q_0 \vec{E}$

$$\therefore \quad \boxed{\Delta V = V_2 - V_1 = -\int_1^2 \vec{E} \cdot d\vec{s}}$$

Note: The integral on the right hand side of the above can be calculated *along any path from point 1 to 2.* (**Path-Independent**)

Convention: $V_\infty = 0 \quad \Rightarrow \quad \boxed{V_P = -\int_\infty^P \vec{E} \cdot d\vec{s}}$

(B) To get $\vec{E}$ from V:



(i.e. Potential = V on the surface)

Again, use the definition of V:

$$\Delta U = q_0 \Delta V = \underbrace{-W}_{\text{Work done}}$$

However,

$$W = \underbrace{q_0 \vec{E}}_{\text{Electric force}} \cdot \Delta\vec{s} = q_0 \, E_s \, \Delta s$$

where E_s is the E-field component along the path $\Delta\vec{s}$.

$$\therefore \quad q_0 \Delta V = -q_0 E_s \Delta s$$

$$\therefore \quad E_s = -\frac{\Delta V}{\Delta s}$$

For infinitesimal Δs,

$$\therefore \quad \boxed{E_s = -\frac{dV}{ds}}$$

Note: (1) Therefore the E-field component along *any direction* is the negtive derivative of the potential *along the same direction.*

(2) If $d\vec{s} \perp \vec{E}$, then $\Delta V = 0$

(3) ΔV is biggest/smallest if $d\vec{s} \parallel \vec{E}$

Generally, for a potential $V(x, y, z)$, the relation between $\vec{E}(x, y, z)$ and V is

$$E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z}$$

$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are **partial derivative**s

For $\frac{\partial}{\partial x} V(x, y, z)$, everything y, z are treated like a *constant* and we only take derivative with respect to x.

Example: If $V(x, y, z) = x^2 y - z$

$$\frac{\partial V}{\partial x} =$$

$$\frac{\partial V}{\partial y} =$$

$$\frac{\partial V}{\partial z} =$$

For other co-ordinate systems

(1) Cylindrical:

$$V(r, \theta, z) \qquad \begin{cases} E_r = -\frac{\partial V}{\partial r} \\ E_\theta = -\frac{1}{r} \cdot \frac{\partial V}{\partial \theta} \\ E_z = -\frac{\partial V}{\partial z} \end{cases}$$

(2) Spherical:

$$V(r,\theta,\phi) \qquad \begin{cases} E_r &= -\dfrac{\partial V}{\partial r} \\[2ex] E_\theta &= -\dfrac{1}{r}\cdot\dfrac{\partial V}{\partial \theta} \\[2ex] E_\phi &= -\dfrac{1}{r\sin\theta}\cdot\dfrac{\partial V}{\partial \phi} \end{cases}$$

Note: Calculating V involves summation of *scalars*, which is easier than adding *vectors* for calculating E-field.

$\therefore$ To find the E-field of a general charge system, we first calculate $\boldsymbol{V}$, and then derive $\vec{\boldsymbol{E}}$ from the partial derivative.

Example: Uniformly charged disk

From potential calculations:

$$V = \frac{\sigma}{2\epsilon_0}\left(\sqrt{R^2+z^2} - |z|\right) \qquad \text{for a point along the z-axis}$$

For $\quad z > 0, \quad |z| = z$

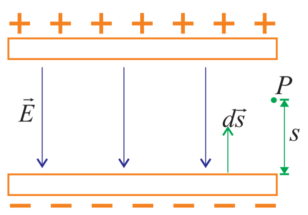
$$\therefore \quad E_z = -\frac{\partial V}{\partial z} = \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right] \qquad \text{(Compare with Chap.2 notes)}$$

Example: Uniform electric field

(`e.g.` Uniformly charged $+ve$ and $-ve$ plates)

Consider a path going from the $-ve$ plate to the $+ve$ plate

Potential at point P, V_P can be deduced from definition.



$$\texttt{i.e.} \quad V_P - V_- = -\int_0^s \vec{E}\cdot d\vec{s} \qquad (V_- = \text{Potential of } -ve \text{ plate})$$

$$= -\int_0^s (-E\,ds) \qquad \because \ \vec{E}, d\vec{s} \text{ pointing opposite directions}$$

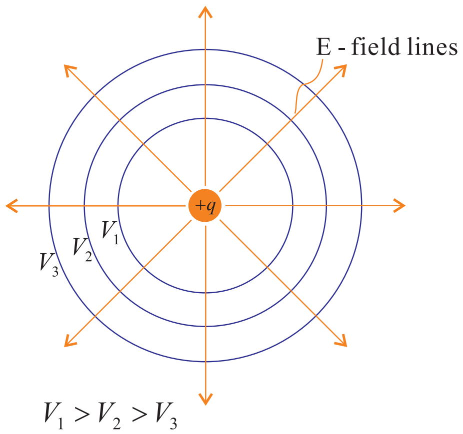
$$= E\int_0^s ds = Es$$

Convenient reference: $\quad V_- = 0$

$$\therefore \quad \boxed{V_P = E\cdot s}$$

4.4 Equipotential Surfaces

Equipotential surface is a surface on which the *potential is constant.*
$\Rightarrow \quad (\Delta V = 0)$



$$V(r) = \frac{1}{4\pi\epsilon_0} \cdot \frac{+q}{r} = const$$

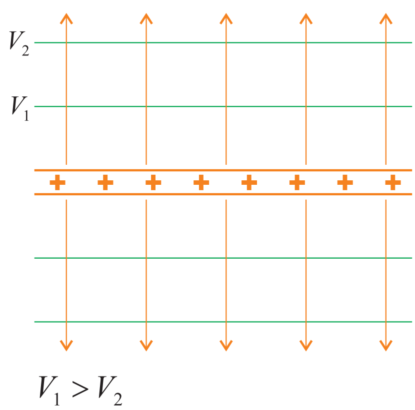
$\Rightarrow \quad r = const$

$\Rightarrow$ Equipotential surfaces are *circles/spherical surfaces*

Note: (1) A charge can move freely on an equipotential surface without any work done.

(2) The **electric field lines** must be *perpendicular* to the **equipotential surfaces**. (Why?)
On an equipotential surface, $V = constant$
$\Rightarrow \Delta V = 0 \Rightarrow \vec{E} \cdot d\vec{l} = 0$, where $d\vec{l}$ is *tangent* to equipotential surface
$\therefore \quad \vec{E}$ must be *perpendicular* to equipotential surfaces.

Example: Uniformly charged surface (infinite)

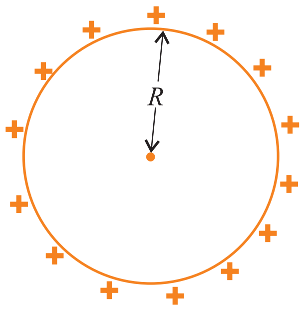


Recall $\quad V = V_0 - \frac{\sigma}{2\epsilon_0}|z|$
$\uparrow$
Potential at $z = 0$

Equipotential surface means

$$V = const \Rightarrow V_0 - \frac{\sigma}{2\epsilon_0}|z| = C$$
$$\Rightarrow |z| = constant$$

Example: Isolated spherical charged conductors



Recall:

(1) E-field inside = 0

(2) charge distributed on the *outside* of conductors.

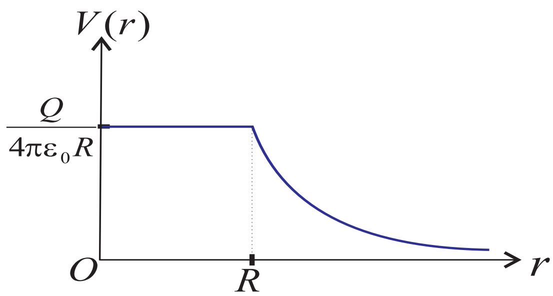
(i) Inside conductor:

$$\begin{aligned} E = 0 \quad &\Rightarrow \quad \Delta V = 0 \text{ everywhere in conductor} \\ &\Rightarrow \quad V = \textit{constant} \text{ everywhere in conductor} \\ &\Rightarrow \quad \text{The entire conductor is at the same potential} \end{aligned}$$

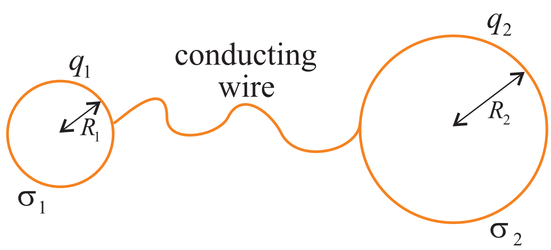
(ii) Outside conductor:

$$V = \frac{Q}{4\pi\epsilon_0 r}$$

$\because$ Spherically symmetric (Just like a point charge.)
BUT not true for conductors of arbitrary shape.



Example: Connected conducting spheres



Two conductors connected can be seen as a *single conductor*

$\therefore$ Potential everywhere is identical.

Potential of radius R_1 sphere $V_1 = \dfrac{q_1}{4\pi\epsilon_0 R_1}$

Potential of radius R_2 sphere $V_2 = \dfrac{q_2}{4\pi\epsilon_0 R_2}$

$$\mathbf{V_1 = V_2}$$

$$\Rightarrow \frac{q_1}{R_1} = \frac{q_2}{R_2} \quad \Rightarrow \quad \frac{q_1}{q_2} = \frac{R_1}{R_2}$$

Surface charge density

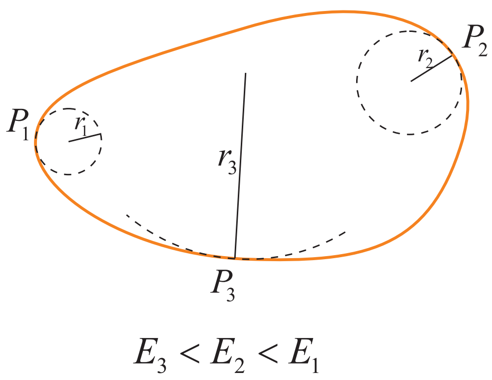
$$\sigma_1 = \underbrace{\frac{q_1}{4\pi R_1^2}}_{\text{Surface area of radius } R_1 \text{ sphere}}$$

$$\therefore \quad \frac{\sigma_1}{\sigma_2} = \frac{q_1}{q_2} \cdot \frac{R_2^2}{R_1^2} = \frac{R_2}{R_1}$$

$$\therefore \quad \text{If } R_1 < R_2, \text{ then } \sigma_1 > \sigma_2$$

And the surface electric field $E_1 > E_2$

For arbitrary shape conductor:



At every point on the conductor, we fit a *circle*. The radius of this circle is the *radius of curvature*.

Note: Charge distribution on a conductor does **not** have to be uniform.

Capacitance and DC Circuits

5.1 Capacitors

A **capacitor** is a system of *two conductors* that carries *equal and opposite charges*. A capacitor *stores charge and energy* in the form of electro-static field.

We define **capacitance** as

$$\boxed{C = \frac{Q}{V}} \qquad \text{Unit: Farad(F)}$$

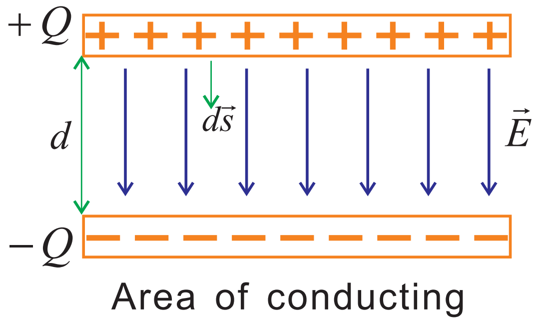
where

$$Q = \text{Charge on one plate}$$
$$V = \text{Potential difference between the plates}$$

Note: The C of a capacitor is a *constant* that depends only on its shape and material.
`i.e.` If we increase V for a capacitor, we can increase Q stored.

5.2 Calculating Capacitance

5.2.1 Parallel-Plate Capacitor



(1) Recall from Chapter 3 note,

$$|\vec{E}| = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$$

(2) Recall from Chapter 4 note,

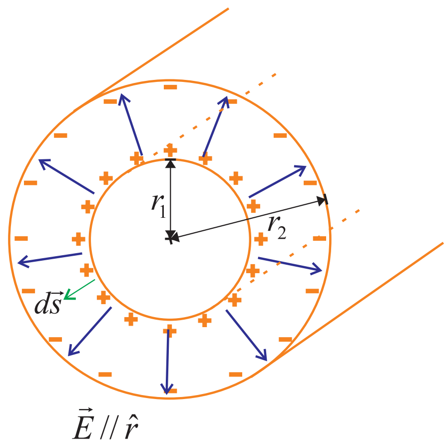
$$\Delta V = V_+ - V_- = -\int_-^+ \vec{E}\cdot d\vec{s}$$

Again, notice that this integral is independent of the path taken.
$\therefore$ We can take the path that is parallel to the $\vec{E}$-field.

$$\begin{aligned} \therefore \quad \Delta V &= \int_+^- \vec{E}\cdot d\vec{s} \\ &= \int_+^- E\cdot ds \\ &= \frac{Q}{\epsilon_0 A}\underbrace{\int_+^- ds}_{\text{Length of path taken}} \\ &= \frac{Q}{\epsilon_0 A}\cdot d \end{aligned}$$

$$(3)\ \therefore \quad \boxed{C = \frac{Q}{\Delta V} = \frac{\epsilon_0 A}{d}}$$

5.2.2 Cylindrical Capacitor



Consider two concentric cylindrical wire of innner and outer radii r_1 and r_2 respectively. The length of the capacitor is L where $r_1 < r_2 \ll L$.

(1) Using Gauss' Law, we determine that the E-field between the conductors is (`cf.` Chap3 note)

$$\vec{E} = \frac{1}{2\pi\epsilon_0} \cdot \frac{\lambda}{r}\,\hat{r} = \frac{1}{2\pi\epsilon_0} \cdot \frac{Q}{Lr}\,\hat{r}$$

where λ is charge per unit length

(2)

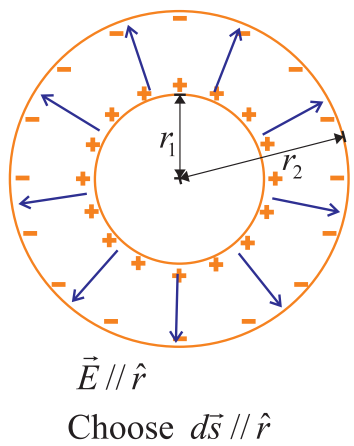
$$\Delta V = \int_{+}^{-} \vec{E} \cdot d\vec{s}$$

Again, we choose the path of integration so that $d\vec{s} \parallel \hat{r} \parallel \vec{E}$

$$\therefore \quad \Delta V = \int_{r_1}^{r_2} E\,dr = \frac{Q}{2\pi\epsilon_0 L} \underbrace{\int_{r_1}^{r_2} \frac{dr}{r}}_{\ln(\frac{r_2}{r_1})}$$

$$\therefore \quad \boxed{C = \frac{Q}{\Delta V} = 2\pi\epsilon_0 \frac{L}{\ln(r_2/r_1)}}$$

5.2.3 Spherical Capacitor



For the space between the two conductors,

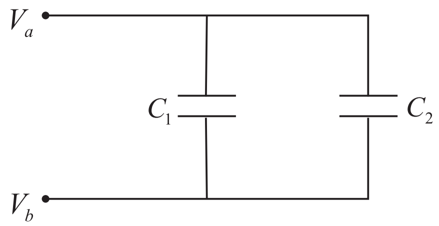
$$E = \frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{r^2}; \qquad r_1 < r < r_2$$

$$\begin{aligned} \Delta V &= \int_{+}^{-} \vec{E} \cdot d\vec{s} \\ \text{Choose } d\vec{s} \parallel \hat{r} \qquad &= \int_{r_1}^{r_2} \frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{r^2}\,dr \\ &= \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] \end{aligned}$$

$$\boxed{C = 4\pi\epsilon_0 \left[\frac{r_1 r_2}{r_2 - r_1}\right]}$$

5.3 Capacitors in Combination

(a) Capacitors in Parallel



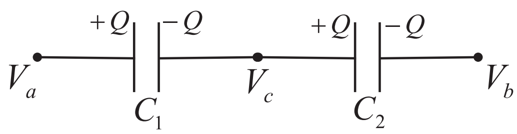
In this case, it's the *potential difference* $V = V_a - V_b$ that is the same across the capacitor.

BUT: *Charge* on each capacitor different

$$\begin{aligned} \text{Total charge} \quad Q &= Q_1 + Q_2 \\ &= C_1 V + C_2 V \\ Q &= \underbrace{(C_1 + C_2)}_{\text{Equivalent capacitance}} V \end{aligned}$$

$\therefore$ For capacitors in parallel: $\boxed{C = C_1 + C_2}$

(b) Capacitors in Series



The *charge across capacitors* are the same.

BUT: *Potential difference* (P.D.) across capacitors different

$$\begin{aligned} \Delta V_1 &= V_a - V_c = \frac{Q}{C_1} \qquad \text{P.D. across } C_1 \\ \Delta V_2 &= V_c - V_b = \frac{Q}{C_2} \qquad \text{P.D. across } C_2 \end{aligned}$$

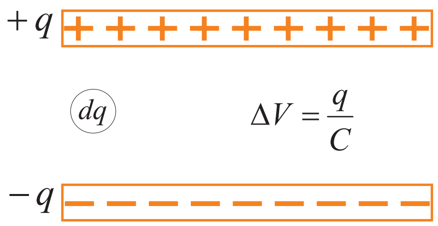
$\therefore$ Potential difference

$$\begin{aligned} \Delta V &= V_a - V_b \\ &= \Delta V_1 + \Delta V_2 \\ \Delta V &= Q\,(\frac{1}{C_1} + \frac{1}{C_2}) = \frac{Q}{C} \end{aligned}$$

where C is the **Equivalent Capacitance**

$$\therefore \quad \boxed{\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}}$$

5.4 Energy Storage in Capacitor



In charging a capacitor, *positive charge* is being moved from the *negative plate* to the *positive plate*.
$\Rightarrow$ NEEDS WORK DONE!

Suppose we move charge dq from $-ve$ to $+ve$ plate, *change in potential energy*

$$dU = \Delta V \cdot dq = \frac{q}{C}\, dq$$

Suppose we keep putting in a total charge Q to the capacitor, the *total potential energy*

$$U = \int dU = \int_0^Q \frac{q}{C}\, dq$$

$$\therefore \quad \boxed{U = \frac{Q^2}{2C} = \frac{1}{2} C\Delta V^2} \qquad (\because Q = C\Delta V)$$

The energy stored in the capacitor is stored in the **electric field** between the plates.

Note : In a parallel-plate capacitor, the *E-field is constant between the plates.*

$\therefore$ We can consider the E-field energy

$$\text{density u} = \frac{\text{Total energy stored}}{\text{Total volume with E-field}}$$

$$\therefore \quad u = \frac{U}{\underbrace{Ad}_{\text{Rectangular volume}}}$$

Recall

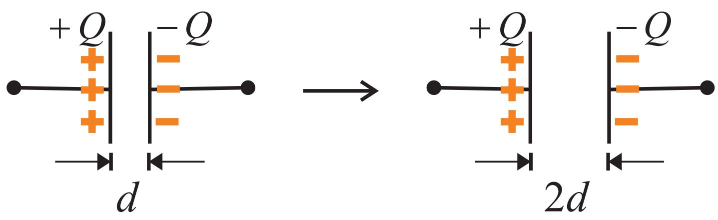
$$\begin{cases} C = \dfrac{\epsilon_0 A}{d} \\ E = \dfrac{\Delta V}{d} \quad \Rightarrow \quad \Delta V = Ed \end{cases}$$

$$\therefore \quad u = \frac{1}{2} \Big(\overbrace{\frac{\epsilon_0 A}{d}}^{C}\Big) \cdot \Big(\overbrace{Ed}^{(\Delta V)^2}\Big)^2 \cdot \overbrace{\frac{1}{Ad}}^{\frac{1}{Volume}}$$

$$\boxed{u = \frac{1}{2}\epsilon_0 E^2}$$ Energy per unit volume of the electrostatic field

$\uparrow$
can be generally applied

Example : Changing capacitance



(1) Isolated Capacitor:
Charge on the capacitor plates remains *constant*.
BUT: $C_{new} = \dfrac{\epsilon_0 A}{2d} = \dfrac{1}{2}C_{old}$
$\therefore \quad U_{new} = \dfrac{Q^2}{2C_{new}} = \dfrac{Q^2}{2C_{old}/2} = 2U_{old}$
$\therefore$ In pulling the plates apart, work done $W > 0$

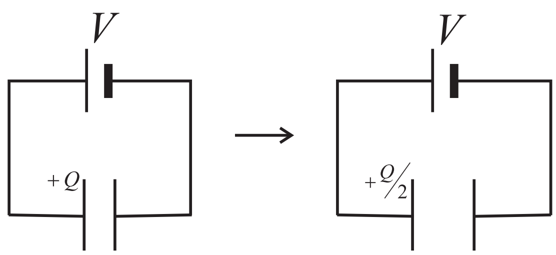
Summary :

	Q	$\rightarrow$	Q	C	$\rightarrow$	$C/2$	
$(V = \frac{Q}{C}) \Rightarrow$	V	$\rightarrow$	$2V$	E	$\rightarrow$	E	$(E = \frac{V}{d})$
$\frac{1}{2}\epsilon_0 E^2 =$	u	$\rightarrow$	u	U	$\rightarrow$	$2U$	$(U = u \cdot volume)$

(2) Capacitor connected to a battery:
Potential difference between capacitor plates remains *constant*.
$U_{new} = \dfrac{1}{2}C_{new}\Delta V^2 = \dfrac{1}{2}\cdot\dfrac{1}{2}C_{old}\Delta V^2 = \dfrac{1}{2}U_{old}$
$\therefore$ In pulling the plates apart, work done by battery < 0

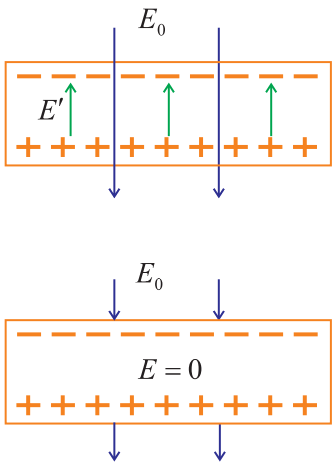
Summary :

Q	$\rightarrow$	$Q/2$	C	$\rightarrow$	$C/2$
V	$\rightarrow$	V	E	$\rightarrow$	$E/2$
u	$\rightarrow$	$u/4$	U	$\rightarrow$	$U/2$



5.5 Dielectric Constant

We first recall the case for a *conductor* being placed in an *external E-field* E_0.



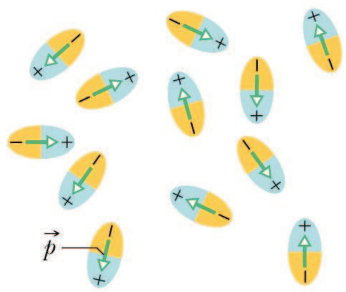
In a conductor, charges are free to move inside so that the *internal E-field* E' set up by these charges

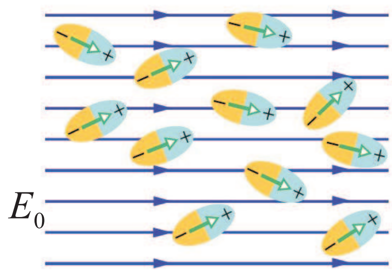
$$E' = -E_0$$

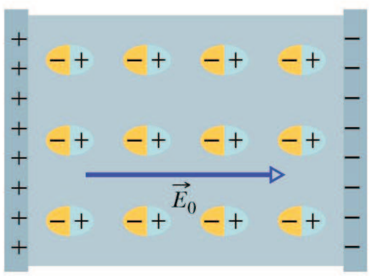
so that E-field inside conductor = 0.

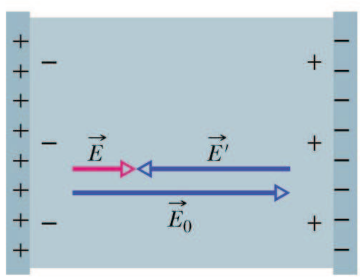
Generally, for **dielectric**, the atoms and molecules behave like a **dipole** in an E-field.

Or, we can envision this so that in the absence of E-field, the *direction of dipole in the dielectric are* randomly distributed.









The aligned dipoles will generate an *induced E-field* E', where $|E'| < |E_0|$.
We can observe the aligned dipoles in the form of *induced surface charge*.

Dielectric Constant : When a dielectric is placed in an external E-field E_0, the E-field inside a dielectric is *induced*.
E-field in dielectric

$$\boxed{E = \frac{1}{K_e} E_0}$$

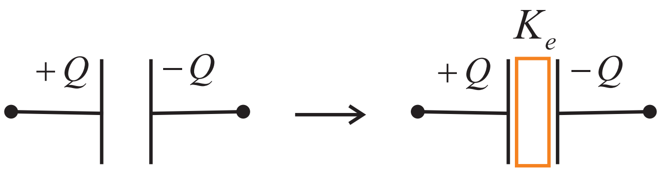
$$K_e \;=\; \textbf{dielectric constant} \quad \geq 1$$

Example :

Vacuum	$K_e = 1$
Porcelain	$K_e = 6.5$
Water	$K_e \sim 80$
Perfect conductor	$K_e = \infty$
Air	$K_e = 1.00059$

5.6 Capacitor with Dielectric

Case I :



Again, the *charge remains constant* after dielectric is inserted.
BUT: $E_{new} = \dfrac{1}{K_e} E_{old}$

$\therefore \quad \Delta V = Ed \quad \Rightarrow \quad \Delta V_{new} = \dfrac{1}{K_e} \Delta V_{old}$

$\therefore \quad C = \dfrac{Q}{\Delta V} \quad \Rightarrow \quad C_{new} = K_e \, C_{old}$

For a parallel-plate capacitor with dielectric:

$$\boxed{C = \frac{K_e \epsilon_0 A}{d}}$$

We can also write $C = \dfrac{\epsilon A}{d}$ in general with

$$\epsilon = K_e\,\epsilon_0 \qquad \text{(called \textbf{permittivity of dielectric})}$$

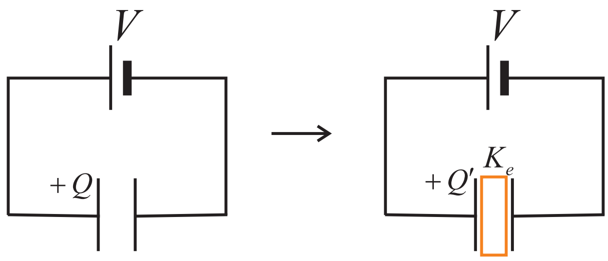
(Recall ϵ_0 = **Permittivity of free space**)

Energy stored $U = \dfrac{Q^2}{2C}$;

$$\therefore \quad U_{new} = \frac{1}{K_e} U_{old} \quad < \quad U_{old}$$

$$\boxed{\therefore \quad \text{Work done in inserting dielectric} < 0}$$

Case II : Capacitor connected to a battery



Voltage across capacitor plates *remains constant* after insertion of dielectric.
In both scenarios, the E-field inside capacitor remains constant
($\because\; E = V/d$)

<u>BUT</u>: How can E-field remain constant?
<u>ANSWER</u>: By having extra charge on capacitor plates.

Recall: For conductors,

$$E = \frac{\sigma}{\epsilon_0} \qquad \text{(Chapter 3 note)}$$

$$\Rightarrow \quad E = \frac{Q}{\epsilon_0 A} \qquad (\sigma = \text{charge per unit area} = Q/A)$$

After insertion of dielectric:

$$E' = \frac{E}{K_e} = \frac{Q'}{K_e \epsilon_0 A}$$

But E-field remains constant!

$$\therefore \quad E' = E \;\Rightarrow\; \frac{Q'}{K_e \epsilon_0 A} = \frac{Q}{\epsilon_0 A}$$

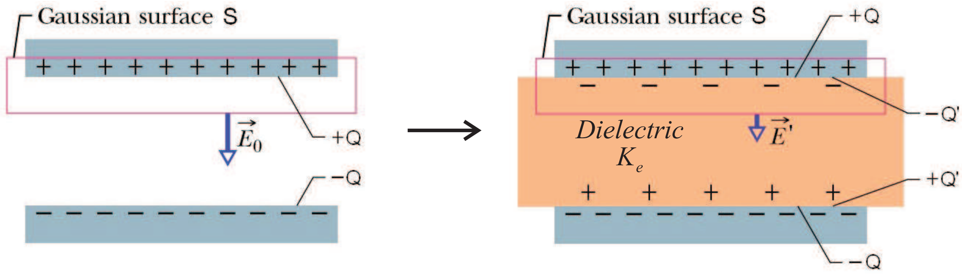
$$\Rightarrow \; Q' = K_e Q \; > \; Q$$

$\therefore$ Capacitor $C = Q/V \Rightarrow C' \to K_e C$
Energy stored $U = \frac{1}{2}CV^2 \Rightarrow U' \to K_e U$
(i.e. $U_{new} > U_{old}$)

$$\boxed{\therefore \text{ Work done to insert dielectric} > 0}$$

5.7 Gauss' Law in Dielectric

The Gauss' Law we've learned is applicable in *vacuum only*. Let's use the capacitor as an example to examine Gauss' Law in dielectric.



Free charge on plates	$\pm Q$	$\pm Q$
Induced charge on dielectric	0	$\mp Q'$

Gauss' Law $\oint_S \vec{E}\cdot d\vec{A} = \frac{Q}{\epsilon_0}$ $\Rightarrow E_0 = \frac{Q}{\epsilon_0 A}$ (1)

Gauss' Law: $\oint_S \vec{E}'\cdot d\vec{A} = \frac{Q-Q'}{\epsilon_0}$ $\therefore E' = \frac{Q-Q'}{\epsilon_0 A}$ (2)

However, we define $E' = \frac{E_0}{K_e}$ (3)

From (1), (2), (3) $\therefore \frac{Q}{K_e\epsilon_0 A} = \frac{Q}{\epsilon_0 A} - \frac{Q'}{\epsilon_0 A}$

$$\therefore \boxed{\text{Induced charge density } \sigma' = \frac{Q'}{A} = \sigma\Big(1 - \frac{1}{K_e}\Big) < \sigma}$$

where σ is free charge density.

Recall Gauss' Law in Dielectric:

$$\epsilon_0 \oint_S \vec{E}'\cdot d\vec{A} = Q - Q'$$

$\uparrow$ E-field in dielectric; $\uparrow$ free charge; $\uparrow$ induced charge

$$\Rightarrow \quad \epsilon_0 \oint_S \vec{E}' \cdot d\vec{A} = Q - Q\Big[1 - \frac{1}{K_e}\Big]$$

$$\Rightarrow \quad \epsilon_0 \oint_S \vec{E}' \cdot d\vec{A} = \frac{Q}{K_e}$$

$$\boxed{\oint_S K_e \vec{E}' \cdot d\vec{A} = \frac{Q}{\epsilon_0}}$$ Gauss' Law in dielectric

Note :

(1) This goes back to the Gauss' Law in vacuum with $E = \dfrac{E_0}{K_e}$ for dielectric

(2) Only *free charges* need to be considered, even for dielectric where there are *induced charges*.

(3) Another way to write:

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon}$$

where $\vec{E}$ is E-field in dielectric, $\epsilon = K_e \epsilon_0$ is Permittivity

Energy stored with dielectric:

Total energy stored: $U = \dfrac{1}{2} CV^2$

With dielectric, recall $C = \dfrac{K_e \epsilon_0 A}{d}$

$V = Ed$

$\therefore$ Energy stored per unit volume:

$$\boxed{u_e = \frac{U}{Ad} = \frac{1}{2} K_e \epsilon_0 E^2}$$

and $u_{dielectric} = K_e u_{vacuum}$

$\therefore$ More energy is stored per unit volume in dielectric than in vacuum.

5.8 Ohm's Law and Resistance

ELECTRIC CURRENT is defined as the flow of electric charge through a cross-sectional area.

$$\boxed{i = \frac{dQ}{dt}}$$ Unit: Ampere (A) = C/second

Convention :

(1) Direction of current is the direction of *flow of positive charge.*

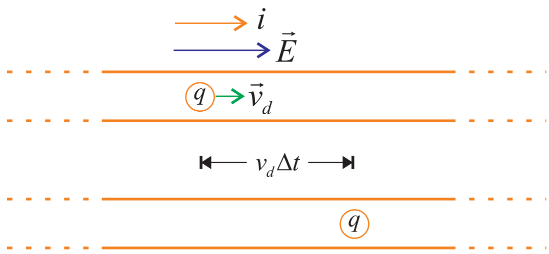
(2) Current is NOT a vector, but the **current density** is a **vector**.

$\vec{j}$ = charge flow per unit time per unit area

$$\boxed{i = \int \vec{j} \cdot d\vec{A}}$$

Drift Velocity :

Consider a current i flowing through a cross-sectional area A:



$\therefore$ In time Δt, total charges passing through segment:

$$\Delta Q = q \underbrace{A(V_d \Delta t)}_{\text{Volume of charge passing through}} n$$

where q is charge of the current carrier, n is density of charge carrier per unit volume

$$\therefore \quad \text{Current:} \quad \boxed{i = \frac{\Delta Q}{\Delta t} = nqAv_d}$$

$$\text{Current Density:} \quad \boxed{\vec{j} = nq\vec{v}_d}$$

Note : For metal, the charge carriers are the free electrons inside.
$\therefore$ $\vec{j} = -ne\vec{v}_d$ for metals
$\therefore$ Inside metals, $\vec{j}$ and $\vec{v}_d$ are in *opposite direction.*

We define a general property, **conductivity** ($\boldsymbol{\sigma}$), of a material as:

$$\boxed{\vec{j} = \boldsymbol{\sigma}\vec{E}}$$

Note : In general, σ is NOT a constant number, but rather a *function of position and applied E-field.*

A more commonly used property, **resistivity** ($\boldsymbol{\rho}$), is defined as $\rho = \dfrac{1}{\sigma}$

$$\therefore \quad \vec{E} = \rho \vec{j}$$

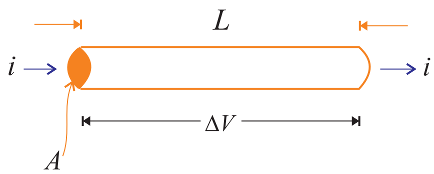
Unit of ρ : Ohm-meter (Ωm)
where Ohm (Ω) = Volt/Ampere

OHM'S LAW:

Ohmic materials have resistivity that are *independent of the applied electric field.*
i.e. metals (in not too high E-field)

Example :

Consider a **resistor** (ohmic material) of length L and cross-sectional area A.



$\therefore$ Electric field inside conductor:

$$\Delta V = \int \vec{E} \cdot d\vec{s} = E \cdot L \quad \Rightarrow \quad E = \frac{\Delta V}{L}$$

Current density: $j = \dfrac{i}{A}$

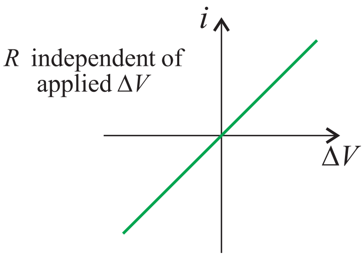
$$\therefore \quad \rho = \frac{E}{j}$$

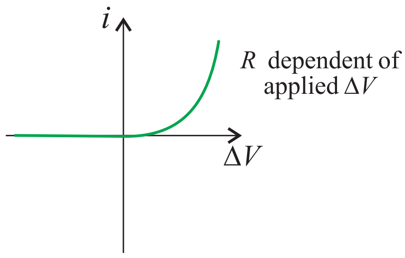
$$\rho = \frac{\Delta V}{L} \cdot \frac{1}{i/A}$$

$$\boxed{\frac{\Delta V}{i} = R = \rho \frac{L}{A}}$$

where R is the **resistance** of the conductor.

Note: $\Delta V = iR$ is NOT a statement of Ohm's Law. It's just a definition for resistance.

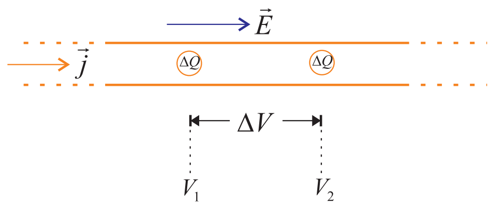




(Read Chap. 29-4 of Halliday Vol 2)

ENERGY IN CURRENT:

Assuming a charge ΔQ enters with potential V_1 and leaves with potential V_2 :



$\therefore$ Potential energy lost in the wire:

$$\Delta U = \Delta Q\, V_2 - \Delta Q\, V_1$$
$$\Delta U = \Delta Q (V_2 - V_1)$$

$\therefore$ Rate of energy lost per unit time

$$\frac{\Delta U}{\Delta t} = \frac{\Delta Q}{\Delta t} (V_2 - V_1)$$

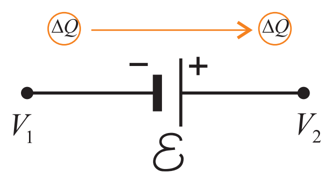
Joule's heating $P = i \cdot \Delta V =$ Power dissipated in conductor

For a resistor R, $P = i^2 R = \dfrac{\Delta V^2}{R}$

5.9 DC Circuits

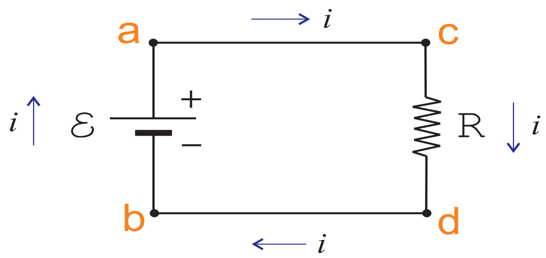
A **battery** is a device that *supplies electrical energy* to maintain a current in a circuit.

In moving from point 1 to 2, electric potential energy increase by $\Delta U = \Delta Q(V_2 - V_1) =$ Work done by $\mathcal{E}$



Define $\mathcal{E} =$ Work done/charge $= V_2 - V_1$

Example :



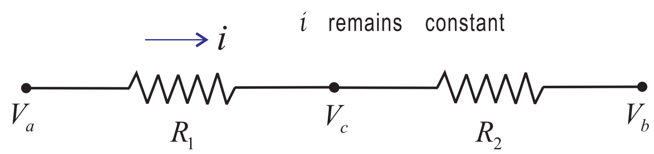
$$\left.\begin{aligned} V_a &= V_c \\ V_b &= V_d \end{aligned}\right\} \quad \text{assuming}^{(1)} \text{ perfect conducting wires.}$$

$$\text{By Definition:} \quad \begin{aligned} V_c - V_d &= iR \\ V_a - V_b &= \mathcal{E} \end{aligned}$$

$$\therefore \quad \mathcal{E} = iR \quad \Rightarrow \quad i = \frac{\mathcal{E}}{R}$$

Also, we have assumed[2] zero resistance inside battery.

Resistance in combination :



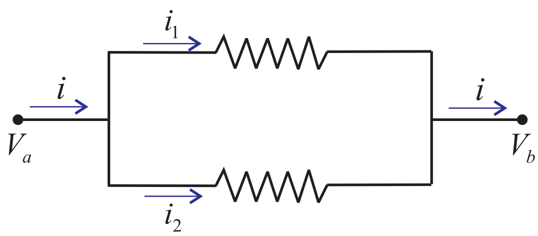
Potential differece (P.D.)

$$\begin{aligned} V_a - V_b &= (V_a - V_c) + (V_c - V_b) \\ &= iR_1 + iR_2 \end{aligned}$$

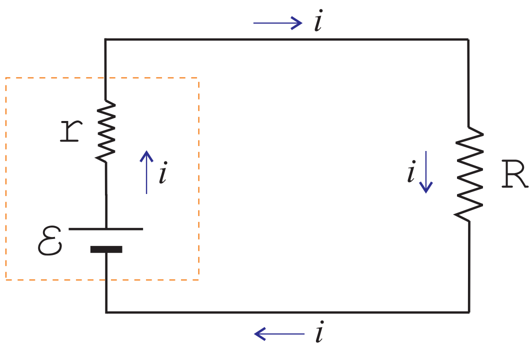
$\therefore$ Equivalent Resistance

$$R = R_1 + R_2 \qquad \text{for resistors in series}$$

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \qquad \text{for resistors in parallel}$$



Example :



For real battery, there is an **internal resistance** that we cannot ignore.

$$\therefore \quad \mathcal{E} = i(R+r)$$
$$i = \frac{\mathcal{E}}{R+r}$$

Joule's heating in resistor R :

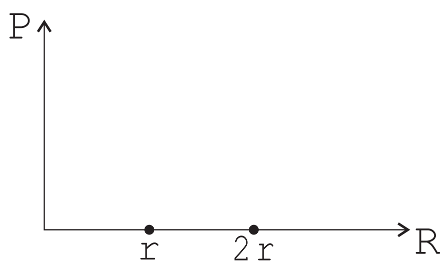
$$P = i \cdot (\text{P.D. across resistor } R)$$
$$= i^2 R$$
$$P = \frac{\mathcal{E}^2 R}{(R+r)^2}$$

Question: What is the value of R to obtain *maximum* Joule's heating?

Answer: We want to find R to *maximize* P.

$$\frac{dP}{dR} = \frac{\mathcal{E}^2}{(R+r)^2} - \frac{\mathcal{E}^2\, 2R}{(R+r)^3}$$

$$\text{Setting} \quad \frac{dP}{dR} = 0 \;\Rightarrow\; \frac{\mathcal{E}^2}{(R+r)^3}\left[(R+r) - 2R\right] = 0$$
$$\Rightarrow\; r - R = 0$$
$$\Rightarrow\; R = r$$

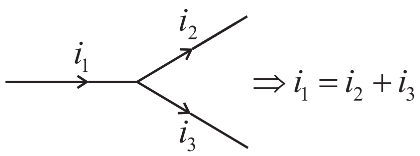


ANALYSIS OF COMPLEX CIRCUITS:

KIRCHOFF'S LAWS:

(1) First Law (Junction Rule):
Total current entering a junction equal to the total current leaving the junction.

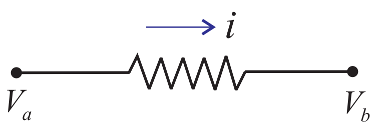
Conservation of charge



(2) Second Law (Loop Rule):
The sum of potential differences around a complete circuit loop is zero.

Convention :

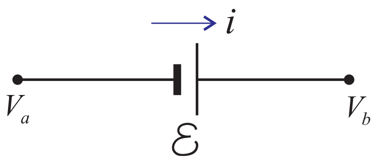
(i)



$$V_a > V_b \quad \Rightarrow \quad \text{Potential difference} = -iR$$

i.e. Potential *drops* across resistors

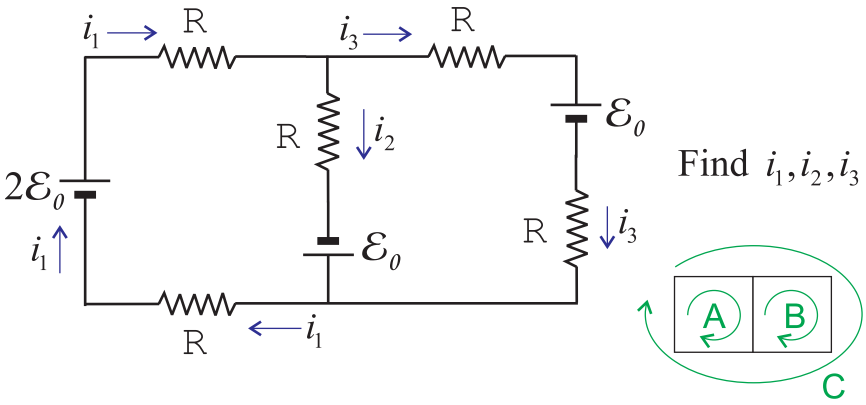
(ii)



$$V_b > V_a \quad \Rightarrow \quad \text{Potential difference} = +\mathcal{E}$$

i.e. Potential *rises* across the negative plate of the battery.

Example :



By junction rule:

$$i_1 = i_2 + i_3 \tag{5.1}$$

By loop rule:

$$\text{Loop A} \;\Rightarrow\; 2\mathcal{E}_0 - i_1 R - i_2 R + \mathcal{E}_0 - i_1 R = 0 \tag{5.2}$$

$$\text{Loop B} \;\Rightarrow\; -i_3 R - \mathcal{E}_0 - i_3 R - \mathcal{E}_0 + i_2 R = 0 \tag{5.3}$$

$$\text{Loop C} \;\Rightarrow\; 2\mathcal{E}_0 - i_1 R - i_3 R - \mathcal{E}_0 - i_3 R - i_1 R = 0 \tag{5.4}$$

BUT: (5.4) = (5.2) + (5.3)
General rule: Need only 3 equations for 3 current

$$i_1 = i_2 + i_3 \tag{5.1}$$

$$3\mathcal{E}_0 - 2i_1 R - i_2 R = 0 \tag{5.2}$$

$$-2\mathcal{E}_0 + i_2 R - 2i_3 R = 0 \tag{5.3}$$

Substitute (5.1) into (5.2) :

$$3\mathcal{E}_0 - 2(i_2 + i_3)R - i_2 R = 0$$

$$\Rightarrow \quad 3\mathcal{E}_0 - 3i_2 R - 2i_3 R = 0 \tag{5.4}$$

Subtract (5.3) from (5.4), `i.e.` (5.4)−(5.3)

$$3\mathcal{E}_0 - (-2\mathcal{E}_0) - 3i_2 R - i_2 R = 0$$

$$\Rightarrow \quad \boxed{i_2 = \frac{5}{4} \cdot \frac{\mathcal{E}_0}{R}}$$

Substitute i_2 into (5.3) :

$$-2\mathcal{E}_0 + \left(\frac{5}{4} \cdot \frac{\mathcal{E}_0}{R}\right) R - 2i_3 R = 0$$

$$\Rightarrow \quad \boxed{i_3 = -\frac{3}{8} \cdot \frac{\mathcal{E}_0}{R}}$$

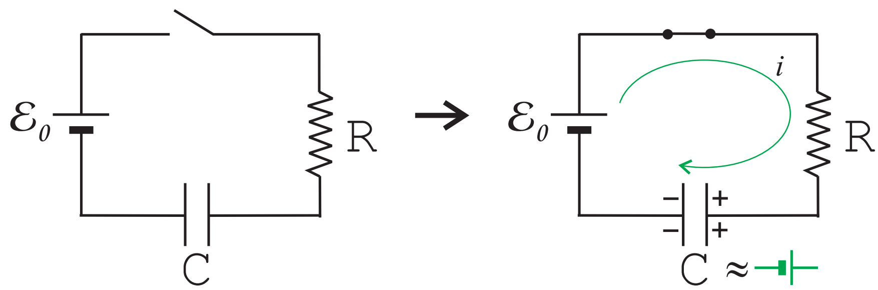
Substitute i_2, i_3 into (5.1) :

$$\boxed{i_1 = \left(\frac{5}{4} - \frac{3}{8}\right)\frac{\mathcal{E}_0}{R} = \frac{7}{8} \cdot \frac{\mathcal{E}_0}{R}}$$

Note: A *negative* current means that it is flowing in *opposite direction* from the one assumed.

5.10 RC Circuits

(A) *Charging* a capacitor with battery:



Using the loop rule:

$$+\mathcal{E}_0 \quad - \underbrace{iR}_{\substack{\text{P.D.}\\ \text{across R}}} \quad - \underbrace{\frac{Q}{C}}_{\substack{\text{P.D.}\\ \text{across C}}} \quad = 0$$

Note: Direction of i is chosen so that the current represents the rate at which the charge on the capacitor is *increasing*.

$$\therefore \quad \mathcal{E} = R\overbrace{\frac{dQ}{dt}}^{i} + \frac{Q}{C} \qquad \text{1st order differential eqn.}$$

$$\Rightarrow \quad \frac{dQ}{\mathcal{E}C - Q} = \frac{dt}{RC}$$

Integrate both sides and use the initial condition:
$t = 0, \quad Q$ on capacitor $= 0$

$$\int_0^Q \frac{dQ}{\mathcal{E}C - Q} = \int_0^t \frac{dt}{RC}$$

$$
\begin{aligned}
& -\ln(\mathcal{E}C - Q)\Big|_0^Q = \frac{t}{RC}\Big|_0^t \\
\Rightarrow\quad & -\ln(\mathcal{E}C - Q) + \ln(\mathcal{E}C) = \frac{t}{RC} \\
\Rightarrow\quad & \ln\Big(\frac{1}{1-\frac{Q}{\mathcal{E}C}}\Big) = \frac{t}{RC} \\
\Rightarrow\quad & \frac{1}{1-\frac{Q}{\mathcal{E}C}} = e^{t/RC} \\
\Rightarrow\quad & \frac{Q}{\mathcal{E}C} = 1 - e^{-t/RC} \\
\Rightarrow\quad & \boxed{Q(t) = \mathcal{E}C(1 - e^{-t/RC})}
\end{aligned}
$$

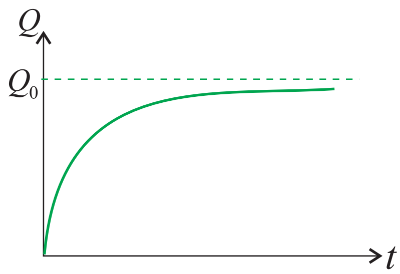
Note: (1) At $t = 0$, $Q(t=0) = \mathcal{E}C(1-1) = 0$

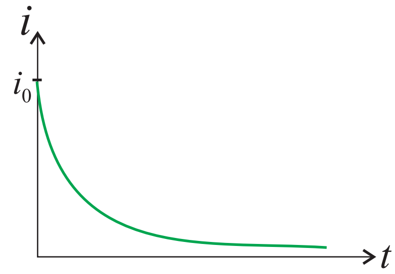
(2) As $t \to \infty$, $Q(t \to \infty) = \mathcal{E}C(1-0) = \mathcal{E}C$ $=$ Final charge on capacitor (Q_0)

(3) Current:

$$
\begin{aligned}
i &= \frac{dQ}{dt} \\
&= \mathcal{E}C\Big(\frac{1}{RC}\Big)e^{-t/RC} \\
i(t) &= \frac{\mathcal{E}}{R}e^{-t/RC}
\end{aligned}
$$

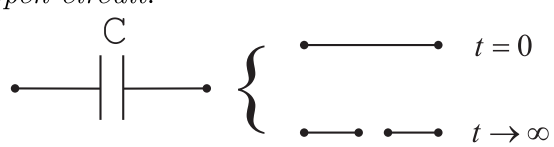
$$
\begin{cases}
i(t=0) = \dfrac{\mathcal{E}}{R} = \text{Initial current} = i_0 \\
i(t\to\infty) = 0
\end{cases}
$$





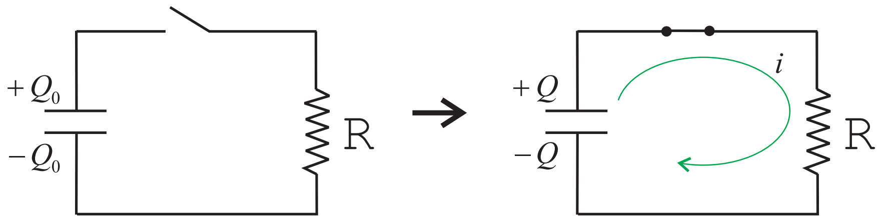
(4) At time $= 0$, the capacitor acts like *short circuit* when there is *zero charge on the capacitor*.

(5) As time $\to \infty$, the capacitor is *fully charged* and current $= 0$, it acts like a *open circuit*.



(6) $\tau_c = RC$ is called the **time constant**. It's the time it takes for the charge to reach $(1 - \frac{1}{e})\, Q_0 \simeq 0.63 Q_0$

(B) *Discharging* a charged capacitor:



Note: Direction of i is chosen so that the current represents the rate at which the charge on the capacitor is *decreasing*.

$$\therefore \quad i = -\frac{dQ}{dt}$$

Loop Rule:

$$V_c - iR = 0$$
$$\Rightarrow \quad \frac{Q}{C} + \frac{dQ}{dt} R = 0$$
$$\Rightarrow \quad \frac{dQ}{dt} = -\frac{1}{RC} Q$$

Integrate both sides and use the initial condition:
$t = 0, \quad Q$ on capacitor $= Q_0$

$$\int_{Q_0}^{Q} \frac{dQ}{Q} = -\frac{1}{RC} \int_0^t dt$$
$$\Rightarrow \quad \ln Q - \ln Q_0 = -\frac{t}{RC}$$
$$\Rightarrow \quad \ln\Big(\frac{Q}{Q_0}\Big) = -\frac{t}{RC}$$
$$\Rightarrow \quad \frac{Q}{Q_0} = e^{-t/RC}$$
$$\Rightarrow \quad Q(t) = Q_0\, e^{-t/RC}$$
$$(i = -\frac{dQ}{dt}) \qquad \Rightarrow \quad i(t) = \frac{Q_0}{RC}\, e^{-t/RC}$$
$$(\text{At } t = 0) \qquad \Rightarrow \quad i(t=0) = \frac{1}{R} \cdot \underbrace{\frac{Q_0}{C}}_{\text{Initial P.D. across capacitor}}$$
$$i_0 = \frac{V_0}{R}$$

At $\quad t = RC = \tau \qquad\qquad Q(t = RC) \;=\; \frac{1}{e}\,Q_0 \;\simeq\; 0.37Q_0$